

E L E M E N T S

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A L G E B R A.

FOR THE USE OF STUDENTS
IN UNIVERSITIES.

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A D V E R T I S E M E N T.

The following Syllabus was drawn up for the use of Students who were to attend the Lectures of the Author ; and is not intended to supersede the perusal of a more compleat System of Algebra.



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INTRODUCTION.

QUANTITY which can be measured, and is the object of mathematics, is of two kinds, *Number* and *Extension*. The former is treated of in *Arithmetic*; the latter in *Geometry*.

Numbers are ranged in a *Scale*, by the continued repetition of some one number, which is called the *Root*; and, in consequence of this order, they are conveniently expressed in words, and denoted by characters. The operations of arithmetic are easily derived from the established method of notation, and the most simple reasonings concerning the relations of magnitude.

Investigations by the common arithmetic are greatly limited, from the want of characters to express the quantities that are unknown, and their different relations to one another, and to such as are known. Hence letters and other convenient symbols have been introduced to supply this defect; and thus gradually has arisen the science of

A

Algebra,

Algebra, properly called UNIVERSAL
ARITHMETIC.

In the common arithmetic too, the given numbers disappear in the course of the operation, so that general rules can seldom be derived from it; but, in algebra, the known quantities, as well as the unknown, may be expressed by letters, which, through the whole operation, retain their original form; and hence may be deduced, not only general canons for like cases, but the dependence of the several quantities concerned, and likewise the determination of a problem, without exhibiting which, it is not completely resolved. This general manner of expressing quantities also, and the general reasonings concerning their connections, which may be founded on it, have rendered this science not less useful in the demonstration of theorems, than in the resolution of problems.

If geometrical quantities be supposed to be divided into equal parts, their relations, in respect of magnitude, or their proportions, may be expressed by numbers; one of these equal parts being denoted by the unit.

unit. Arithmetic, however, is used in expressing only the conclusions of geometrical propositions ; and it is by algebra, that the bounds and application of geometry have been of late so far extended.

The proper objects of mathematical science, are NUMBER and EXTENSION; but mathematical inquiries may be instituted also concerning any physical quantities, that are capable of being measured or expressed by numbers and extended magnitudes : And, as the application of algebra may be equally universal, it has been called *The science of quantity in general.*

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ELEMENTS

OF

ALGEBRA.

PART I.

DEFINITIONS.

I. **Q**UANTITIES which are known, are generally represented by the first letters of the alphabet, as a, b, c , &c. and such as are unknown, by the last letters, as x, y, z , &c.

II. The sign $+$ (*plus*), denotes, that the quantity before which it is placed, is to be added. Thus $a+b$, denotes the sum of a and b ; $3+5$ denotes the sum of 3 and 5, or 8. When no sign is expressed, $+$ is understood.

III.

III. The sign $-$ (*minus*) denotes, that the quantity, before which it is placed, is to be subtracted. Thus, $a-b$ denotes the excess of a above b ; $6-2$ is the excess of 6 above 2, or 4.

Note, These characters $+$ and $-$ from their extensive use in Algebra, are called *the signs*; and the one is said to be *opposite* or *contrary* to the other.

IV. Quantities which have the sign $+$ prefixed to them, are called *positive* or *affirmative*; and such as have the sign $-$ prefixed to them, are called *negative*.

V. Quantities which have the same sign, either $+$ or $-$, are also said to have *like signs*, and those which have different signs, are said to have *unlike signs*. Thus, $+a, +b$, have like signs, and $+a, -c$, are said to have unlike signs.

VI. The *juxtaposition* of letters as in the same word, expresses the product of the quantities denoted by these letters. Thus ab expresses the product of a and b ; bcd expresses the continued product of b, c , and d . The sign $\times$ also expresses

presses the product of any two quantities between which it is placed.

VII. A number prefixed to a letter, is called a *numeral coefficient*, and expresses the product of the quantity by that number, or how often the quantity denoted by the letter is to be taken. When no number is prefixed, unit is understood.

VIII. The *quotient* of two quantities is denoted, by placing the *dividend* above a small line, and the *divisor* below it.

Thus, $\frac{18}{3}$ is the quotient of 18 divided by

3, or 6; $\frac{a}{b}$ is the quotient of a divided by b . This expression of a quotient is also called a *fraction*.

IX. A quantity is said to be *simple*, which consists of one part or *Term*, as $+a$, $-abc$; and a quantity is said to be *compound*, when it consists of more than one term, connected by the signs $+$ or $-$. Thus, $a+b$, $a-b+c$, are compound quantities. If there are two terms, it is called a *binomial*; if three, a *trinomial*, &c.

X.

X. Simple quantities, or the terms of compound quantities, are said to be *like*, which consist of the same letter or letters, equally repeated. Thus $+ab$, $-5ab$, are like quantities; but $+ab$, and $+qab$, are unlike.

XI. The equality of two quantities is expressed, by placing the sign $=$ between them. Thus $x+a=b-c$, means that the sum of x and a , is equal to the excess of b above c .

When quantities are considered abstractly, then $+$ and $-$ denote addition and subtraction only, according to Def. II. III. and the terms *positive* and *negative* express the same ideas. In that case, a negative quantity by itself is unintelligible. The sign $+$ also is unnecessary before simple quantities, or before the leading term of a compound quantity which is not negative; though, when such a quantity or term is to be added to another, $+$ must be placed before it, to express that addition; and hence in Def. II. it is said, that $+$ is understood, when no sign is expressed.

In

In geometry, however, and in certain applications of geometry and algebra, there may be an opposition or contrariety in the quantities, analogous to that of addition and subtraction ; and the signs + and — may very conveniently be used to express that contrariety. In such cases, negative quantities are understood to exist by themselves; and the same rules take place in operations into which they enter, as are used with regard to the negative terms of abstract quantities.

When, therefore, in the following elementary rules, negative quantities are introduced as examples, they are to be understood, either as subtracted from positive quantities not expressed ; or, as having such an opposition as has now been mentioned, to other quantities marked with the sign

Of Axioms.

ALGEBRAICAL reasonings are founded on a set of first principles, not less evident than the axioms of geometry. They are indeed so simple, that it

B

is.

is unnecessary; in this place, either to enumerate, or to illustrate them. Besides, a general axiom is not more evident than any particular proposition comprehended in it.

CHAP.

C H A P. I.

Fundamental Operations.

THE fundamental operations in Algebra are the same as in common arithmetic; ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION; and from the various combinations of these four, all the others are derived.

P R O B. I. *To add Quantities.*

Simple quantities, or the terms of compound quantities, to be added together, may be *like with like signs, like with unlike signs, or they may be unlike.*

CASE I. To add terms that are like, and have like signs.

Rule. *Add together the coefficients, to their sum prefix the common sign, and subjoin the common letter or letters.*

EXAMPLES.

E X A M P L E S.

To	$5ab$	$3aa-ab$
Add	$4ab$	$7aa-2ab$
		$4aa-5ab$
Sum	$9ab$	
		$14aa-8ab.$

CASE II. To add Terms that are like, but have unlike signs.

Rule. Subtract the less coefficient from the greater; prefix the sign of the greater to the remainder, and subjoin the common letter or letters.

E X A M P L E S.

$-4a$	$+7bc$	$-5ab$
$+7a$	$-3bc$	$+2ab$
	$+bc$	$+3ab$
$+3a$		
	$+5bc$	0

CASE III. To add Terms that are unlike.

Rule. Set them all down, one after another, with their signs and coefficients prefixed.

X A M P L E.

$$\begin{array}{r}
 2a + 3b \\
 -5c + 8 \\
 \hline
 2a + 3b - 5c + 8
 \end{array}$$

Compound

Compound quantities are added together, by uniting the several terms, of which they consist, by the preceding rules.

E X A M P L E.

$$\begin{array}{r} \text{The sum of } \left\{ \begin{array}{l} 5ab - 3xy - 12cd \\ 7xy - ab + 15 \\ 9cd - xy - mn \end{array} \right. \\ \hline \text{is } 4ab - 3cd + 15 - mn \end{array}$$

The rule for case III. may be considered as the general rule for adding all Algebraical Quantities whatsoever, and, by the rules in the two preceding cases, the like Terms in the quantities to be added may be united, so as to render the expression of the sum more simple.

P R O B. II. *To subtract Quantities.*

General Rule. *Change the signs of the quantity to be subtracted into the contrary signs, and then add it, so changed, to the quantity from which it was to be subtracted: (by Prob. I.) the sum arising by this addition, is the remainder.*

E X A M P L E S.

EXAMPLES.

From	$+5a$	$7ab-16bc$
Subtract	$+3a$	$3ab+mb$
	$+2a$	$4ab-16bc-mb$
Rem.		

From	$5a-7b+9c+8$
Subt.	$2a-4b+9c-d$
	$3a-3b+8+d$
Rem.	

When a positive quantity is to be subtracted, the rule is obvious from Def. 3. : In order to shew it, when the negative part of a quantity is to be subtracted, let $c-d$ be subtracted from a , the remainder, according to the rule, is $a-c+d$. For, if c is subtracted from a , the remainder is $a-c$, (by Def. 3.) ; but this is too small, because c is subtracted instead of $c-d$, which is less than it by d ; the remainder therefore is too small by d ; and d being added, it is $a-c+d$, according to the rule.

Otherwise, If the quantity d be added to these two quantities a , and $c-d$, the difference will continue the same; that is, the excess of a above $c-d$, is equal to the excess of $a+d$ above $c-d+d$; that is, to the excess of $a+d$ above c , which plainly is

a

$a+d-c$, and is therefore the remainder required.

PROB. III. *To multiply Quantities.*

General Rule for the signs. *When the signs of the two terms to be multiplied are like, the sign of the product is + ; but, when the signs are unlike, the sign of the product is —.*

CASE I. *To multiply two terms.*

Rule. *Find the sign of the product by the general rule ; after it place the product of the numeral coefficients, and then set down all the letters one after another, as in one word.*

Mult.	$+a$	$+5b$	$-5ax$
By	$+b$	$-3c$	$-7ab$
	$+ab$	$-15bc$	$+35aabc$

The reason of this rule is derived from Def. 6. and from the nature of multiplication, which is a repeated addition of one of the quantities to be multiplied as often as there are units in the other. Hence also the

the letters in two terms multiplied together may be placed in any order, and therefore the order of the alphabet is generally preferred.

CASE II. To multiply compound quantities.

Rule. *Multiply every term of the multiplicand by all the terms of the multiplier, one after another, according to the preceding rule, and then collect all the products into one sum; that sum is the product required.*

EXAMPLES.

Mult.	$2a+3b$	$m+x$
By	$3ax-4by$	$m-x$
	$6aax+9abx$	$mm+mx$
	$-8aby-12bby$	$-mx-xx$
Prod.	$6aax+9abx-8aby-12bby$	$mm+mx-xx$

Mult.	$a-b$
By	$c-d$
	$ac-cb$
	$-ad+db$
Prod.	$ac-cb-ad+db$

of

Of the general Rule for the Signs.

The reason of that rule will appear, by proving it, as applied to the last mentioned example of $a-b$ multiplied by $c-d$, in which every case of it occurs.

Since multiplication is a repeated addition of the multiplicand, as often as there are units in the multiplier, hence, if $a-b$ is to be multiplied by c , $a-b$ must be added to itself, as often as there are units in c , and the product therefore must be $ca-cb$. (Prob. I.)

But this product is too great; for $a-b$ is to be multiplied, not by c , but by $c-d$ only, which is the excess of c above d ; d times $a-b$, therefore, or $da-db$, has been taken too much; hence, this quantity must be subtracted from the former part of the product, and the remainder, which (by prob. II.) is $ca-cb-da+db$, will be the true product required.

Def. XII. When several Quantities are multiplied together, any of them is called a *factor* of the Product.

XIII. The products arising from the continual multiplication of the same quantity, are called the *Powers* of that quantity, which is the *Root*. Thus, aa , aaa , $aaaa$, &c. are *powers* of the *root* a .

XIV. These powers are expressed, by placing above the root, to the right hand, a figure, denoting how often the root is repeated. This figure is called an *index* or *exponent*, and from it the power is denominated. Thus,

$$\left. \begin{array}{l} a \\ aa \\ aaa \\ aaaa \end{array} \right\} \begin{array}{l} \text{is called} \\ \text{the} \\ \text{is called} \end{array} \left\{ \begin{array}{l} 1^{\text{st}} \\ 2^{\text{d}} \\ 3^{\text{d}} \\ 4^{\text{th}} \end{array} \right\} \begin{array}{l} \text{Power of the Root} \\ a, \text{ and is o-} \\ \text{therwise expres-} \\ \text{sed by} \end{array} \left\{ \begin{array}{l} a^1 \text{ or } a \\ a^2 \\ a^3 \\ a^4 \text{ \&c.} \end{array} \right.$$

The 2d and 3d powers are generally called the *Square* and *Cube*; and the 4th, 5th, and 6th, are also sometimes respectively called the *Biquadrate*, *Sur-solid*, and *Cubocube*.

Cor. Powers of the same root are multiplied by adding their exponents. Thus, $a^3 \times a^2 = a^5$, or $aaa \times aa = aaaaa$, $b^3 \times b = b^4$.

SCHOLIUM.

S C H O L I U M.

Sometimes it is convenient to express the multiplication of quantities, by setting them down with the sign ($\times$) between them, without performing the operation according to the preceding rules; thus $a^2 \times b$ is written instead of a^2b ; and $\overline{a-b} \times \overline{c-d}$, expresses the product of $a-b$, multiplied by $c-d$.

Def. XV. A *Vinculum* is a line drawn over any number of terms of a compound quantity, to denote those which are understood to be affected by the particular sign connected with it.

Thus, in the last example, it shews, that the terms $+a$ and $-b$, and also c and $-d$ are all affected by the sign ($\times$). Without the vinculum, the expression $a-b \times c-d$, would mean the excess of a above bc and d ; and $\overline{a-b} \times c-d$ would mean the excess of the Product of $a-b$ by c , above d . Thus also, $\overline{a+b}^2$, expresses the second power of $a+b$, or the product of that quantity multiplied by itself; whereas $a+b^2$ would express

press only the sum of a and b^2 ; and so of others. By some writers a *parenthesis* () is used as a vinculum, & $(a+b)^2$ is the same thing as $\overline{a+b}^2$.

P R O B. IV. *To divide quantities.*

General rule for the signs. If the signs of the divisor and dividend are like, the sign of the quotient is +; if they are unlike, the sign of the quotient is —.

This rule is easily deduced from that given in Prob. 3.; for, from the nature of division, the quotient must be such a quantity as, multiplied by the divisor, shall produce the dividend with its proper sign.

From Def. 8. the Quotient of any two Quantities may be expressed, by placing the dividend above a line, and the divisor below it. But a quotient may often be expressed in a more simple and convenient form, as will appear from the following distinction of the Cases.

Case I. When the divisor is simple, and is a *factor* of all the terms of the Dividend. This is easily discovered by inspection; for

for then the coefficient of the divisor measures that of all the Terms of the dividend, and all the letters of the divisor are found in every Term of the Dividend.

Rule. The letter or letters in the divisor are to be expunged out of each term in the dividend, and the coefficients of each term to be divided by the coefficient of the divisor ; the quantity resulting is the quotient.

Ex. a) $ab(b. \quad 2aab) \quad 6a^3bc - 4a^2bdc \quad (3ac - 2dm$

The reason of this is evident from the nature of division, and from def. 6.

Note. It is obvious from corollary to prob. 3. that powers of the same root are divided by subtracting their exponents.

Thus $a^2)a^3 (a \quad a^3)a^7 (a^4. \quad \text{also } a^2b)a^3b^6 (ab^5.$

CASE II. When the divisor is simple, but not a factor of the dividend.

Rule. The quotient is expressed by a fraction, according to def. 8. viz. by placing the

the dividend above a line, and the divisor below it.

Thus the quotient of $3ab^2$ divided by $2mbc$ is the fraction $\frac{3ab^2}{2mbc}$.

Such expressions of quotients may often be reduced to a more simple form, as shall be explained in the second part of this chapter.

CASE III. When the divisor is compound.

R U L E.

1. *The terms of the dividend are to be ranged according to the powers of some one of its letters ; and those of the divisor, according to the powers of the same letter.*

Thus, if $a^2+2ab+b^2$ is the dividend, and $a+b$ the divisor, they are ranged according to the powers of a .

2. *The first term of the dividend is to be divided by the first term of the divisor, (observing*

serving the general rule for the signs;) and this quotient being set down as a part of the quotient wanted, is to be multiplied by the whole divisor, and the product subtracted from the dividend. If nothing remain, the division is finished: the remainder, when there is any, is a new dividend.

Thus, in the preceding example, a^2 divided by a , gives a , which is the first part of the quotient wanted: and the product of this part by the whole divisor $a+b$, viz. a^2+ab being subtracted from the given dividend, there remains in this example $ab+b^2$.

3. *Divide the first term of this new dividend by the first term of the divisor as before, and join the quotient to the part already found, with its proper sign: then multiply the whole divisor by this part of the quotient, and subtract the product from the new dividend: and thus the operation is to be continued, till no remainder is left, or till it appear, that there will always be a remainder.*

Thus,

Thus, in the preceding example, $+ab$, the first term of the new dividend divided by a , gives b ; the product of which, multiplied by $a+b$, being subtracted from $ab+b^2$, nothing remains, and $a+b$ is the true quotient.

The entire operation is as follows.

$$\begin{array}{r}
 a+b \overline{) a^2+2ab+b^2} \quad (a+b \\
 \underline{a^2+ab} \\
 ab+b^2 \\
 \underline{ab+b^2} \\
 0
 \end{array}$$

$$\begin{array}{r}
 3a-b \overline{) 3a^3-12a^2-a^2b+10ab-2b^2} \quad (a^2-4a+2b \\
 \underline{3a^3} \\
 -12a^2 \\
 \underline{-12a^2} \\
 0 \\
 +6ab-2b^2 \\
 \underline{+6ab-2b^2} \\
 0
 \end{array}$$

$$\begin{array}{r}
 1-a \overline{) 1+a+a^2+a^3, \&c.} \\
 \underline{1-a} \\
 +a \\
 \underline{+a-a^2} \\
 +a^2 \\
 \underline{+a^2-a^3} \\
 +a^3, \&c.
 \end{array}$$

It

It often happens, as in the last example, that there is still a remainder from which the operation may be continued without end. This expression of a quotient is called *an infinite series*; the nature of which shall be considered afterwards. By comparing a few of the first terms, the *law* of the series may be discovered, by which, without any more division, it may be continued to any number of terms wanted.

Of the General Rule.

The reason of the different parts of this rule is evident; for, in the course of the operation, all the terms of the quotient obtained by it, are multiplied by all the terms of the divisor, and the products are successively subtracted from the dividend, till nothing remain: that, therefore, from the nature of division, must be the true quotient.

Note. The sign $\div$ is sometimes used to express the quotient of two quantities, between which it is placed: Thus,

$\overline{a^2+x^2} \div \overline{a+x}$, expresses the quotient of a^2+x^2 divided by $a+x$.

D

C H A P,

C H A P. I.

P A R T II.

OF F R A C T I O N S.

D E F I N I T I O N S.

- I. **W**HEN a quotient is expressed by a fraction, the dividend above the line is called the *numerator*; and the divisor below it is called the *denominator*.
- II. If the numerator is less than the denominator, it is called a *proper fraction*.
- III. If the numerator is not less than the denominator, it is called an *improper fraction*.

IV.

IV. If one part of a quantity is an integer, and the other a fraction, is called a *mixt quantity*.

V. The *reciprocal* of a fraction, is a fraction whose numerator is the denominator of the other ; and whose denominator is the numerator of the other. The reciprocal of an integer is the quotient of 1 divided by that integer. Thus, $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; and $\frac{1}{m}$ is the reciprocal of m .

The distinctions in Def. II. III. IV. properly belong to common arithmetic, from which they are borrowed, and are scarcely used in Algebra.

The operations concerning fractions are founded on the following proposition :

If the divisor and dividend be either both multiplied, or both divided, by the same quantity, the quotient is the same ; or, if both the numerator and denominator of the fraction be either multiplied or divided by the same

same quantity, the value of that fraction is the same.

Thus, let $\frac{a}{b}=c$, then $\frac{ma}{mb}=c$. For, from the nature of division, if the quotient $\frac{a}{b}$ ($=c$) be multiplied by the divisor b , the product must be the dividend a . Hence $(\frac{a}{b} \times b)_{bc=a}$, and likewise $ma=mbc$, and dividing both by mb , $\frac{ma}{mb}=c$. Conversely, if $\frac{ma}{mb}=c$, then also $\frac{a}{b}=c$.

Cor. 1. Hence a fraction may be reduced to another of the same value, but of a more simple form, by dividing both numerator and denominator by any common measure.

$$\text{Thus, } \frac{30ax-54ay}{12ab} = \frac{5x-9y}{2b}$$

$$\frac{8ab+6ac}{4a^2} = \frac{4b+3c}{2a}$$

Cor. 2. A fraction is multiplied by any integer, by multiplying the numerator, or dividing the denominator by that integer : and conversely, a fraction is divided by any integer,

integer, by dividing the numerator, or multiplying the denominator by that integer.

Prob. I. To find the greatest common measure of two quantities.

1. Of pure numbers.

Rule. Divide the greater by the less: and, if there is no remainder, the less is the greatest common measure required. If there is a remainder, divide the last divisor by it; and thus proceed continually dividing the last divisor by its remainder, till no remainder is left, and the last divisor is the greatest common measure required.

The greatest common measure of 45 and 63 is 9; the greatest common measure of 187 and 391 is 17. Thus,

$$\begin{array}{r} 45 \overline{) 63} \quad (1 \\ \underline{45} \end{array}$$

$$\begin{array}{r} 18 \overline{) 45} \quad (2 \\ \underline{36} \end{array}$$

$$\begin{array}{r} 9 \overline{) 18} \quad (2 \\ \underline{18} \end{array}$$

$$\begin{array}{r} 187 \overline{) 391} \quad (2 \\ \underline{374} \end{array}$$

$$\begin{array}{r} 17 \overline{) 187} \quad (11 \\ \underline{187} \end{array}$$

*

From

From the nature of this operation it is plain, that it may always be continued, till there be no remainder. The rule depends on the two following principles.

1. A quantity which measures both divisor and remainder, must measure the dividend.

2. A quantity which measures both divisor and dividend, must also measure the remainder.

For a quantity which measures two other quantities, must also measure both their sum and difference; and, from the nature of division, the dividend consists of the divisor repeated a certain number of times, together with the remainder. By the first it appears, that the number found by this rule is a common measure; and, by the second, it is plain there can be no greater common measure; for, if there were, it must necessarily measure the quantity already found less than itself, which is absurd.

When the greatest common measure of algebraical quantities is required, if either of them be simple, any common simple
divisor

divisor is easily found by inspection. If they are both compound, any common simple divisor may also be found by inspection. But, when the greatest compound divisor is wanted, the preceding rule is to be applied ; only,

2. *The simple divisors of each of the quantities are to be taken out, the remainders in the several operations are also to be divided by their simple divisors, and the quantities are always to be ranged according to the powers of the same letter.*

The simple divisors in the given quantities, or in the remainders, do not affect a compound divisor which is wanted; and hence also, to make the division succeed, any of the dividends may be multiplied by a simple quantity. Besides the simple divisors in the remainders not being found in the divisors from which they arise, can make no part of the common measure sought; and for the same reason, if in such a remainder, there be any compound divisor which does not measure the divisor from which it proceeds, it may be taken out.

EXAMPLES,

EXAMPLES.

$$\begin{array}{r} a^2 - b^2 \) \ a^2 - 2ab + b^2 \ (1 \\ \underline{a^2 - b^2} \end{array}$$

$-2ab + 2b^2$ Remainder which
divided by $-2b$ is $a - b$ $\begin{array}{r} a^2 - b^2 \ (a + b \\ \underline{a^2 - b^2} \end{array}$

* *

If the quantities given are $8a^2b^2 - 10ab^3 + 2b^4$,
and $9a^4b - 9a^3b^2 + 3a^2b^3 - 3ab^4$. The
simple divisors being taken out, viz.
 $2b^2$ out of the first, it becomes
 $4a^2 - 5ab + b^2$, and $3ab$ out of the second, it
is $3a^3 - 3a^2b + ab^2 - b^3$. As the latter is to be
divided by the former, it must be multipli-
ed by 4, to make the operation succeed, and
then it is as follows.

$$\begin{array}{r} 4a^2 - 5ab + b^2 \) \ 12a^3 - 12a^2b + 4ab^2 - 4b^3 \ (3a \\ \underline{12a^3 - 15a^2b + 3ab^2} \\ 3a^2b + ab^2 - 4b^3 \end{array}$$

This remainder is to be divided by b ,
and the new dividend multiplied by 3, to
make the division proceed. Thus,

$$\begin{array}{r} 3a^2+ab-4b^2 \quad 12a^2-15ab+3b^2 \quad (4) \\ 12a^2+4ab-16b^2 \\ \hline \end{array}$$

$$-19ab+19b^2$$

and this remainder, divided by $-19b$, gives $a-b$, which being made a divisor, divides $3a^2+ab-4b^2$ without a remainder, and therefore $a-b$ is the greatest compound divisor: but there is a simple divisor b , and therefore $a-b \times b$ is the greatest common measure required.

Prob. II. To reduce a fraction to its lowest terms.

Rule. Divide both numerator and denominator by their greatest common measure, which may be found by prob. I.

Thus, $\frac{75abc}{125bcx} = \frac{3a}{5x}$, $25bc$ being the greatest common

measure, $\frac{a^4-b^4}{a^5-a^3b^2} = -\frac{a^2+b^2}{a^3}$, also,

$\frac{9a^4b-9a^3b^2+3a^2b^3-3ab^4}{8a^2b^2-10ab^3+2b^4} = \frac{9a^3+3db^2}{8ab-b^2}$ the greatest common measure being $a-b \times b$, by prob. I.

Prob. III. To reduce an integer to the form of a fraction.

E

Rule.

Rule. Multiply the given integer by any quantity for a numerator, and set that quantity under the product, for a denominator.

$$\text{Thus, } a = \frac{ma}{m}, \quad a+b = \frac{a^2-b^2}{a-b}.$$

Cor. Hence, in the following operations concerning fractions, an integer may be introduced; for, by this problem, it may be reduced to the form of a fraction. The denominator of an integer is generally made 1.

Prob. IV. To reduce fractions with different denominators to fractions of equal value, that shall have the same denominator.

Rule. Multiply each numerator, separately taken into all the denominators but its own, and the products shall give the new numerators. Then multiply all the denominators into one another, and the product shall give the common denominator.

Example.

Example. Let the fractions be $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$; they are respectively equal to $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$, $\frac{bde}{bdf}$.

The reason of the operation appears from the preceding proposition; for the numerator and denominator of each fraction are multiplied by the same quantities; and the value of the fractions therefore is the same.

Prob. V. To add and subtract fractions.

Rule. *Reduce them to a common denominator, then add or subtract the numerators; and the sum or difference set over the common denominator is the sum or remainder required.*

Ex. Add together $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, the sum is $\frac{adf+cbf+bde}{bdf}$.

From $\frac{a}{b}$ subtr. $\frac{c}{d}$ the difference is $\frac{ad-bc}{bd}$.

From the nature of division it is evident, that, when several quantities are to be divided by the same divisor, the sum of the quotients is the same with the quotient of the sum of the quantities divided by that common divisor.

In

In like manner the difference of two fractions having the same denominator, is equal to the difference of the numerators divided by that common denominator.

Cor. 1. By Cor. Prob. 3. Integers may be reduced to the form of fractions, and hence integers and fractions may be added and subtracted by this rule. Hence also what is called a mixt quantity may be reduced into the form of a fraction by bringing the integral part into the form of a fraction, with the same denominator as the fractional part, and adding or subtracting the numerators according as the two parts are connected by the signs + or —.

$$\text{Thus, } b + \frac{c}{d} = \frac{bd+c}{d}; \text{ and } a - \frac{a^2-b^2}{2a} = \frac{2a^2 - a^2 + b^2}{2a} = \frac{a^2+b^2}{2a}$$

Cor. 2. A fraction, whose numerator is a compound quantity, may be distinguished into parts, by dividing the numerator into several parts, and setting each over the original denominator, and uniting the new fractions,

fractions, (reduced if necessary), by the signs of their numerators.

$$\text{Thus, } \frac{a^2 - 2ab + b^2}{2a} = \frac{a^2}{2a} - \frac{2ab}{2a} + \frac{b^2}{2a} = \frac{a}{2} - b + \frac{b^2}{2a}.$$

Prob. VI. To multiply fractions.

Rule. Multiply their numerators into one another, to obtain the numerator of the product; and the denominators, multiplied into one another, shall give the denominator of the product.

$$\text{Ex. } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a+b}{c} \times \frac{a-b}{d} = \frac{a^2 - b^2}{cd}$$

For, if $\frac{a}{b}$ is to be multiplied by c , the product is $\frac{ca}{b}$; but if it is to be multiplied only by $\frac{c}{d}$, the former product must be divided by d , and it becomes $\frac{ca}{bd}$ (cor. 2. to the preceding problem.)

Or, let $\frac{a}{b} = m$, and $\frac{c}{d} = n$. Then $a = bm$, and $c = dn$, and $ac = bdmn$, and $(mn =) \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Prob. VII. To divide fractions.

Rule. Multiply the numerator of the dividend by the denominator of the divisor; their

their product shall give the numerator of the quotient. Then multiply the denominator of the dividend by the numerator of the divisor, and their product shall give the denominator.

Or, Multiply the dividend by the reciprocal of the divisor; the product will be the quotient wanted.

$$\text{Thus, } \frac{a}{b} \bigg) \frac{c}{d} \left(\frac{bc}{ad} = \frac{c}{d} \times \frac{b}{a} \right).$$

For, if $\frac{c}{d}$ is to be divided by a , the quotient is $\frac{c}{da}$; but $\frac{c}{d}$ is to be divided, not by a , but by $\frac{a}{b}$; therefore the former quotient must be multiplied by b , and it is $\frac{bc}{da}$.

Or, let $\frac{a}{b} = m$, and $\frac{c}{d} = n$; then $a = bm$, and $c = dn$; also $ad = bdm$ and $bc = bdn$; therefore $\left(\frac{bdn}{bdm} = \right)$
 $\frac{n}{m} = \frac{bc}{ad}$.

SCHOLIUM.

By these problems, the four fundamental operations may be performed, when any
 terms

terms of the original quantities, or of those which arise in the course of the operation, are fractional.

Example

$$\text{Mult. } \frac{a^2}{2x} - \frac{3ax}{2b}$$

$$\text{By } \frac{ab}{3x} - 4x$$

$$\text{Prod. } \frac{a^3b}{6x^2} - \frac{a^2}{2} - 2a^2 + \frac{6ax^2}{b}$$

$$a+x) \frac{a^2+x^2}{a^2+ax} (a-x+ \frac{2x^2}{a} - \frac{2x^3}{a^2} \&c.$$

$$\begin{array}{r} -ax+x^2 \\ -ax-x^2 \\ \hline \end{array}$$

$$\begin{array}{r} 2x^2 \\ 2x^2+2x^3 \\ \hline \end{array}$$

$$a$$

$$\begin{array}{r} -\frac{2x^3}{a} \\ \hline \end{array}$$

$$\begin{array}{r} -\frac{2x^3}{a} - \frac{2x^4}{a^2} \\ \hline \end{array}$$

$$+ \frac{2x^4}{a^2} \&c.$$

This quotient becomes a series, of which the law of continuation is obvious, without any farther operation.

In

In such cases, when we arrive at a remainder of one term, it is commonly set down with the divisor below it, after the other terms of the quotient, which then becomes a mixt quantity.

Thus the last quotient is also expressed by

$$a-x + \frac{2x^3}{a+x}$$

CHAP.

C H A P. II.

Of Proportion.

BY the preceding operations quantities of the same kind may be compared together.

The relation arising from this comparison is called *Ratio* or *Proportion*, and is of two kinds. If we consider the difference of the two quantities, it is called *Arithmetical Proportion*; and if we consider their quotient, it is called *Geometrical Proportion*. This last being most generally useful, is commonly called simply *Proportion*.

1. *Of Arithmetical Proportion.*

Definition. When of four quantities the difference of the first and second is equal to the difference of the third and fourth, the quantities are called *arithmetical proportionals*.

Cor. Three quantities may be arithmetically proportional, by supposing the two middle terms of the four to be equal.

F

Prop.

Prop. In four quantities arithmetically proportional, the sum of the extremes is equal to the sum of the means.

Let the four be a, b, c, d . Therefore from Def. $a - b = c - d$; to these add $b + d$ and $a + d = b + c$.

Cor. 1. Of four arithmetical proportionals, any three being given, the fourth may be found.

Thus, let a, b, c , be the 1st, 2d, and 4th terms, and let x be the 3d which is sought.

Then by def. $a + c = b + x$, and $x = a + c - b$.

Cor. 2. If three quantities be arithmetical proportionals, the sum of the extremes is double of the middle term; and hence, of three such proportionals, any two being given, the third may be found.

2. Of Geometrical Proportion.

Definition. If of four quantities, the quotient of the first and second is equal to the quotient of the third and fourth, these quantities are said to be in *geometrical proportion*. They are also called *proportionals*.

Thus,

Thus, if a, b, c, d , are the four quantities, then $\frac{a}{b} = \frac{c}{d}$. and their ratio is thus denoted, $a : b :: c : d$.

Cor. Three quantities may be geometrical proportionals, viz. by supposing the two middle terms of the four to be equal. If the quantities are a, b, c , then $\frac{a}{b} = \frac{b}{c}$, and the proportion is expressed thus, $a : b : c$.

Prop. I. The product of the extremes of four quantities geometrically proportional is equal to the product of the means : and conversely.

Let $a : b :: c : d$.

Then by Def. $\frac{a}{b} = \frac{c}{d}$

and multiplying both by bd , $ad = bc$.

If $ad = bc$, then dividing by bd , $\frac{a}{b} = \frac{c}{d}$, that is $a : b :: c : d$.

Cor. 1. The product of the extremes of three quantities, geometrically proportional, is equal to the square of the middle term.

Cor.

Cor. 2. Of four quantities geometrically proportional, any three being given, the fourth may be found.

Ex. Let a, b, c , be the three first; to find the 4th. Let it be x , then $a : b :: c : x$, and by this proposition,

$$ax = bc$$

and dividing both by a , $x = \frac{bc}{a}$.

This coincides with the *Rule of Three* in arithmetic, and may be considered as a demonstration of it. In applying the rule to any particular case, it is only to be observed, that the quantities must be so connected and so arranged, that they be proportional, according to the preceding definition.

Cor. 3. Of three geometrical proportionals, any two being given, the third may be found.

Prop. II. If four quantities be geometrically proportional, then if any equimultiples whatever be taken of the first and third, and also any equimultiples whatever of the second and fourth; if the multiple of the first be greater than that of the second, the multiple

tiple of the third will be greater than that of the fourth ; and if equal, equal ; and if less, less.

For, let a, b, c, d , be the four proportionals. Of the first and third, ma and mc may represent any equimultiples whatever, and also, nb, nd , may represent any equimultiples of the second and fourth. Since $a : b :: c : d$, $ad = bc$; and hence multiplying by mn , $mnad = mnbc$, and therefore (Conv. Prop. 1.) $ma : nb :: mc : nd$; and from the definition of proportionals, it is plain, that if ma is greater than nb , mc must be greater than nd ; and if equal, equal ; and if less, less.

Prop. III. If four quantities are proportionals, they will also be proportionals when taken *alternately* or *inversely*, or by *composition*, or by *division*, or by *conversion*. See def. 13. 14. 15. 16. 17. of book V. of Euclid, Simson's edition.

By Prop. II. they will also be proportionals according to Def. 5. book V. of Euclid ; and therefore this proposition is demonstrated

demonstrated by Propositions 16, B, 18, 17, E of the same book.

Otherwise algebraically.

Let $a : b :: c : d$, and therefore $ad = bc$.

Altern. $a : c :: b : d$

Invert. $b : a :: d : c$

Divid. $a - b : b :: c - d : d$

Comp. $a + b : b :: c + d : d$

Convert. $a : a - b :: c : c - d$

For since $ad = bc$, it is obvious, that in each of these cases the product of the extremes is equal to the product of the means; the quantities are therefore proportionals. (prop. 1.)

Prop. IV. If four numbers be proportionals, according to Def. 5. V. B. of Euclid, they will be geometrically proportional, according to the preceding definition.

1st, Let the four numbers be integers, and let them be a, b, c, d . Then if b times a and b times c be taken, and also a times b and a times d , since ba the multiple of the first is equal to ab the multiple of the second, bc , the multiple

tiple

tiple of the third, must be equal to ad the multiple of the fourth. And since $bc = ad$, by prop. 1. a, b, c , and d must be geometrical proportionals.

2dly, If any of the numbers be fractional, all the four being multiplied by the denominators of the fractions, they continue proportionals, according to Def. 5. B. V. Euclid. (by prop. IV. of that book.) and the four integer quantities produced being such proportionals, they will be geometrical proportionals, by the first part of this prop.; and therefore, being reduced by division to their original form, they manifestly will remain proportionals, according to the algebraical definition.

CHAP.

C H A P. III.

Of Equations in general, and of the Solution of simple Equations.

DEFINITIONS.

- I. **A**N *Equation* may in general be defined to be a proposition asserting the equality of two quantities; and is expressed by placing the sign = between them.
- II. When a quantity stands alone upon one side of an equation, the quantities on the other side are said to be a *value* of it. Thus in the equation $x = b + y - d$, x stands alone on one side, and $b + y - d$ is a value of it.
- III. When an unknown quantity is made to stand alone on one side of an equation,
- G and

and there are only known quantities on the other; that equation is said to be *resolved*; and the value of the unknown quantity is called a *root* of the equation.

IV. Equations containing only one unknown quantity and its powers, are divided into *orders*, according to the highest power of the unknown quantity to be found in any of its terms.

If the highest power of the unknown quan- tity in any term be the	}	1st, 2d, 3d, &c.	}	The Equati- on is called	{	Simple, Quadrat. Cubic, &c.
---	---	------------------------	---	-----------------------------	---	-----------------------------------

But the exponents of the unknown quantity are supposed to be integers, and the equation is supposed to be cleared of fractions, in which the unknown quantity, or any of its powers, enter the denominators.

Thus, $x + a = \frac{3x-b}{c}$ is a simple equation ;

$3x - \frac{5}{2x} = 12$, when cleared of the fraction by multiplying both sides by $2x$, becomes $6x^2 - 5 = 24x$ a quadratic. $x^3 - 2x^4 = x^6 - 20$ is an equation of the 6th order, &c.

As

As the general relations of quantity which may be treated of in Algebra, are almost universally either that of equality, or such as may be reduced to that of equality, the doctrine of equations becomes one of the chief branches of the science.

The most common and useful application of Algebra, is in the investigation of quantities that are unknown, from certain given relations to each other, and to such as are known; and hence it has been called the *Analytical Art*. The equations employed for expressing these relations must therefore contain one or more unknown quantities, and the principal business of this art will be, the deducing equations containing only one unknown quantity, and resolving them.

The solution of the different orders of equations will be successively explained; the preliminary rules in the following section are useful in all orders, and are alone sufficient for the solution of simple equations.

SECT.

S E C T. I.

Of simple Equations, and their Resolution.

Simple equations are resolved by the four fundamental operations already explained; and the application of them to this purpose is contained in the following rules.

Rule I. *Any quantity may be transposed from one side of an Equation to the other, by changing its sign.*

Thus, if $3x - 10 = 2x + 5$

Then, $3x - 2x = 10 + 5$ or $x = 15$

Thus also, $5x + b = a + 2x$

By transp. $3x = a - b$.

This rule is obvious from prob. 1. and 2.; for it is equivalent to adding equal quantities to both sides of the equation, or to subtracting equal quantities from both sides.

Cor. The signs of all the terms of an equation

equation may be changed into the contrary signs, and it will continue to be true.

Rule II. *Any quantity by which the unknown quantity is multiplied may be taken away, by dividing all the other quantities of the Equation by it.*

Thus, if $ax = b$

$$x = \frac{b}{a}$$

Also, if $mx + nb = am$

$$x + \frac{nb}{m} = a$$

For if equal quantities are divided by the same quantity, the quotients are equal.

Rule III. *If a term of an Equation is fractional, its denominator may be taken away, by multiplying all the other terms by it.*

Thus, if $\frac{x}{a} = b + c$

$$x = ab + ac$$

Also, if $a - \frac{b}{x} = c$

$$ax - b = cx$$

And by transf. $ax - cx = b$

$$\text{And by div. } x = \frac{b}{a-c}$$

For if all the terms of the equation are multiplied

multiplied by the same quantity, it remains a true proposition.

Corollary to the three last rules.

If any quantity be found on both sides of the equation, with the same sign, it may be taken away from both. (Rule I.)

Also, if all the terms in the equation are multiplied or divided by the same quantity, it may be taken out of them all. (Rule II. and III.)

Ex. If $3x + a = a + b$, then $3x = b$.

If $2ax + 3ab = ma + a^2$, then $2x + 3b = m + a$.

If $\frac{x}{3} - \frac{4}{3} = \frac{16}{3}$ then $x - 4 = 16$.

Any simple equation may be resolved by these rules in the following manner. *1st*, Any fractions may be taken away by R. 3. *2dly*, All the terms including the unknown quantity, may be brought to one side of the equation, and the known terms to the other by R. 1. *Lastly*, If the unknown quantity is multiplied by any known quantity, it may be made to stand alone

alone by R. 2. and the equation will then be resolved. Def. 3.

Examples of simple Equations resolved by these rules.

I.

$$\text{If } 3x + 5 = x + 9$$

$$\text{R. 1. } 2x = 4$$

$$\text{R. 2. } x = \frac{4}{2} = 2$$

II.

$$\text{If } 5x - \frac{5x}{2} + 12 = \frac{4x}{3} + 26$$

$$\text{R. 1. } 5x - \frac{5x}{2} - \frac{4x}{3} = 14$$

$$\text{R. 3. } 30x - 15x - 8x = 84$$

$$\text{Or } 7x = 84$$

$$\text{R. 2. } x = \frac{84}{7} = 12$$

III.

$$\text{If } \frac{5}{x} + \frac{9}{4} = 16$$

$$\text{R. 3. } \frac{20}{x} + 9 = 64$$

$$\text{R. 3. } 20 + 9x = 64x$$

$$\text{R. 1. } 20 = 55x$$

$$\text{R. 2. } x = \frac{20}{55} = \frac{4}{11}$$

SECT. II. *Solution of Questions producing simple Equations.*

From the resolution of equations we obtain the resolution of a variety of useful problems, both in pure mathematics and physics, and also in the practical arts founded upon these sciences. In this place, we consider the application of it to those questions where the quantities are expressed by numbers, and their magnitude alone is to be considered.

When an equation, containing only one unknown quantity, is deduced from the question by the following rules, it is sometimes called a *final Equation*. If it be simple, it may be resolved by the preceding rules, but if it be of a superior order, it must be resolved by the rules afterwards to be explained. The examples in this chapter are so contrived, that the final equation may be simple.

The rules given in this section, for the solution of questions, though they contain

a reference to simple equations only, are to be considered as general, and as applicable to questions which produce equations of any order.

GENERAL RULE.

The unknown quantities in the question proposed must be expressed by letters, and the relations of the known and unknown quantities contained in it, or the conditions of it, as they are called, must be expressed by equations. These equations being resolved by the rules of this science, will give the answer of the question.

For example, if the question is concerning two numbers, they may be called x and y , and the conditions from which they are to be investigated, must be expressible by equations.

Thus, if it be required, that
 the sum of two numbers
 sought be 60, that condition is expressed thus

$$\left. \begin{array}{l} \text{the sum of two numbers} \\ \text{sought be 60, that condition} \\ \text{is expressed thus} \end{array} \right\} x + y = 60$$

If their difference must be 24,

then

$$x - y = 24$$

If their product is 1640, then

$$xy = 1640$$

If their quotient must be 6,

then

$$\left\{ \begin{array}{l} x \\ y \end{array} \right\} = 6$$

If their ratio is as 3 to 2, then

$x : y :: 3 : 2$, and therefore

$$\left\{ \begin{array}{l} 2x \\ 3y \end{array} \right\}$$

H

These

These are some of the relations, which are most easily expressed : Many others occur, which are less obvious ; but as they cannot be described in particular rules, the algebraical expression of them is best explained by examples, and must be acquired by experience.

A distinct conception of the nature of the question, and of the relations of the several quantities to which it refers, will generally lead to the proper method of stating it, which in effect may be considered only as a translation from common language, into that of Algebra.

Case I. When there is only one unknown quantity to be found.

Rule. An equation involving the unknown quantity must be deduced from the question (by the general rule). This equation being resolved by the rules of the last section, will give the answer.

It is obvious, that, when there is only one unknown quantity, there must be only one independent equation contained in the question ;

question; for any other would be unnecessary, and might be contradictory to the former.

EXAMPLE I.

To find a number, to which if there be added, a half, a third part, and a fourth part of itself, the sum will be 50.

Let it be z : then half of it is $\frac{z}{2}$, a third of it $\frac{z}{3}$ &c.

Therefore, $z + \frac{z}{2} + \frac{z}{3} + \frac{z}{4} = 50$

$$24z + 12z + 8z + 6z = 1200$$

$$50z = 1200$$

$$z = 24.$$

If the operation be more complicated, it may be useful to register the several steps of it, as in the following

EXAMPLE II.

A trader allows 100 l. per annum for the expences of his family, and augments yearly that part of his stock which is not so expended, by a third of it; at the end of three years, his original stock was doubled: What had he at first?

Let

Lethis first stock be	1	z
Of which he		
spends the first	2	$z - 100$
year 100 l. and		
there remains		
This remainder	3	$z - 100 + \frac{z - 100}{3} = \frac{4z - 400}{3}$
is increased by a		
third of itself		
The second year	4	$\frac{4z - 400}{3} - 100 = \frac{4z - 700}{3}$
he spends 100 l.		
& there remains		
He increases the	5	$\frac{4z - 700}{3} + \frac{4z - 700}{9} = \frac{16z - 2800}{9}$
remainder by		
one third of it		
The third year he	6	$\frac{16z - 2800}{9} - 100 = \frac{16z - 3700}{9}$
spends 100 l. and		
there remains		
He increases it	7	$\frac{16z - 3700}{9} + \frac{16z - 3700}{27} = \frac{64z - 14800}{27}$
by one third		
But at the end		
of the third year	8	$\frac{64z - 14800}{27} = 2z$
his stock is dou-		
bled; therefore		
By R. 3.	9	$64z - 14800 = 54z$
By R. 1.	10	$10z = 14800$
By R. 2.	11	$z = 1480$

Therefore his stock was 1480 l. which being tried, answers the conditions of the question.

Case II. When there are two unknown quantities.

Rule. *Two independent equations involving the two unknown quantities, must be derived from the question. A value of one of*

of the unknown quantities must be derived from each of the equations : and these two values being put equal to each other, a new equation will arise, involving only one unknown quantity, and may therefore be resolved by the preceding rule.

Two equations must be deduced from the question ; for, from one including two unknown quantities, it is plain, a known value of either of them cannot be obtained ; more than two equations would be unnecessary, and if any third condition were assumed at pleasure, most probably it would be inconsistent with the other two, and a question containing three such conditions would be absurd.

It is to be observed, however, that the two conditions, and hence the two equations expressing them, must be independent, that is, the one must not be deducible from the other by any algebraical reasoning ; for, otherwise, there would in effect be only one equation, under two different forms, from which no solution can be derived.

E X.

E X A M P L E III.

Two persons, A and B, were talking of their ages; says A to B, Seven years ago I was just three times as old as you were, and seven years hence I shall be just twice as old as you will be: I demand their present ages.

Let the ages of	1	x and y
A and B be re-		
spectively		
Seven years ago	2	$x - 7$ and $y - 7$
they were		
Seven years hence	3	$x + 7$ and $y + 7$
they will be		
Therefore by Quest.		
1. and 2.	4	$x - 7 = 3 \times y - 7 = 3y - 21$
also by Quest.		
and 3.	5	$x + 7 = 2 \times y + 7 = 2y + 14$
By 4. & transp.	6	$x = 3y - 14$
By 5. & transp.	7	$x = 2y + 7$
By 6. and 7.	8	$3y - 14 = 2y + 7$
Transp. & 8.	9	$y = 21$
By 9 & 6. or 7.	10	$x = 49$

The ages of A and B then are 49 and 21, which answer the conditions.

The operation might have been a little shortened, by subtracting the 4th from 5th, and thus $14 = -y + 35$; and hence $y = 21$. therefore (by 6th) $x = (3y - 14) = 49$.

E X A M P L E IV.

A gentleman distributing money among some poor people, found he wanted 10 s. to be able to give 5 s. to each; therefore he gives each 4 s. only, and finds he has 5 s. left.—To find the number of shillings and poor people.

If any question such as this, in which there are two quantities sought, can be resolved by means of one letter, the solution is in general more simple than when two are employed. There must be, however, two independent conditions, one of which is used in the notation of one of the unknown quantities, and the other gives an equation.

Let the number of poor be	$11z$
The number of shillings will be	$25z - 10$
The number of shillings is also	$34z + 5$
By 2. and 3.	$4z - 10 = 4z + 5.$
Transp.	$5z = 15.$

The number of poor therefore is 15, and the number of shillings is $(4z + 5 =) 65$, which answer the conditions.

E X A M P L E V.

A courier sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and

and 8 hours after, another sets out from the same place, and travels the same road, at the rate of 5 miles in 3 hours: I demand how long and how far the first must travel, before he is overtaken by the second?

Let the number of hours	1	y
which the first travelled be	2	$y - 8$
Then the second travelled	3	$(5 : 7 :: y :) \frac{7y}{5}$ miles
The first travelled seven	4	$(3 : 5 :: y - 8 :) \frac{5y - 40}{3}$ miles
miles in 5 hours, and	5	$\frac{5y - 40}{3} = \frac{7y}{5}$
therefore in y hours	6	$25y - 200 = 21y$
In like manner the second	7	$4y = 200$
travelled in $y - 8$ hours	8	$y = 50$
But they both travelled the		
same number of miles ;		
therefore by 3. and 4.		
Mult.		
Transp.		
Divid.		

The first then travelled 50 hours, the second ($y - 8 =$) 42 hours.

The miles travelled by each ($\frac{7y}{5} = \frac{5y - 40}{3} =$) 70.

Case III. When there are three or more unknown quantities.

Rule. When there are three unknown quantities, there must be three independent equations arising from the question; and from each of these a value of one of the unknown quantities must be obtained. By comparing

comparing these three values, two equations will arise, involving only two unknown quantities, which may therefore be resolved by the rule for case 2.

In like manner may the rule be extended to such questions as contain four or more unknown quantities; and hence it may be inferred, That, when just as many independent equations may be derived from a question, as there are unknown quantities in it, these quantities may be found by the resolution of equations.

E X A M P L E VI.

To find three numbers, so that the first, with half the other two, the second with one third of the other two, and the third with one fourth of the other two, may each be equal to 34.

I

Let

Let the numbers be x, y, z , and the equations are

	1	$x + \frac{y+z}{2} = 34$
	2	$y + \frac{x+z}{3} = 34$
	3	$z + \frac{x+y}{4} = 34$
From the 1st,	4	$x = \frac{68 - y - z}{2}$
From the 2d,	5	$x = 102 - 3y - z$
From the 3d,	6	$x = 136 - 4z - y$
From 4th and 5th	7	$\frac{68 - y - z}{2} = 102 - 3y - z$
7th reduced	8	$y = \frac{136 - z}{5}$
5 = 6, and reduced	9	$y = \frac{3z - 34}{2}$
8 and 9	10	$\frac{3z - 34}{2} = \frac{136 - z}{5}$
10th reduced	11	$15z - 170 = 272 - 2z$
	12	$17z = 442$ or $z = 26$
By 8 and 5	13	$y = 22$ and $x = 10$

EXAMPLE VII.

To find a number consisting of three places, whose digits are in arithmetical proportion; if this number be divided by the sum of its digits, the quotient will be 48; and if from the number be subtracted 198, the digits will be inverted.

Let the 3 digits be	1	x, y, z
Then the number is	2	$100x + 10y + z$
If the digits be inverted, it is	3	$100z + 10y + x$
The digits are in ar. prop. therefore	4	$x + z = 2y$
By question	5	$\frac{100x + 10y + z}{x + y + z} = 48$
By question	6	$100x + 10y + z - 198 = 100z + 10y + x$
From 6 and transf.	7	$99x = 99z + 198$
Divid. by 99	8	$x = z + 2$
From 4	9	$x = 2y - z$
8 and 9	10	$2y - z = z + 2$
Transp.	11	$y = z + 1$
Mult. 5.	12	$100x + 10y + z = 48x + 48y + 48z$
Transp.	13	$52x = 38y + 47z$
8 and 11 substit. for x and y	14	$52z + 104 = 38z + 38 + 47z$
Transf.	15	$33z = 66$
Divid.	16	$\begin{cases} z = 2 \\ y = (z + 1) = 3 \\ x = (z + 2) = 4. \end{cases}$

The number then is 432, which succeeds upon trial.

It sometimes happens, that all the unknown quantities, when there are more than two, are not in all the equations expressing the conditions, and therefore the preceding rule cannot be literally followed. The solution, however, will be obtained by such substitutions as are used in Ex. 7. and 9. or by similar operations, which need not be particularly described.

Corollary to the preceding rules.

It appears that, in every question, there must be as many independent equations as unknown quantities; if there are not, then the question is called *indeterminate*, because it may admit of an infinite number of answers; since the equations wanting may be assumed at pleasure. There may be other circumstances, however, to limit the answers to one, or a precise number, and which, at the same time, cannot be directly expressed by equations. Such are these; that the numbers must be integers, squares, cubes, and many others. The solution of such problems, which are also
called

called *Diophantine*, shall be considered afterwards.

SCHOLIUM.

On many occasions, by particular contrivances, the operations by the preceding rules may be much abridged. This, however, must be left to the skill and practice of the learner. A few examples are the following.

1. It is often easy to employ fewer letters than there are unknown quantities, by expressing some of them from a simple relation to others contained in the conditions of the question. Thus, the solution becomes more easy and elegant. (See Ex. 4. 5.)

2. Sometimes it is convenient to express by letters, not the unknown quantities themselves, but some other quantities connected with them, as their sum, difference, &c. from which they may be easily derived. (See Ex. 1. of chap. 5.)

3. In the operation also, circumstances will suggest a more easy road than that pointed

pointed out by the general rules. Two of the original equations may be added together, or may be subtracted; sometimes they must be previously multiplied by some quantity, to render such addition or subtraction effectual, in exterminating one of the unknown quantities, or otherwise promoting the solution. Substitutions may be made of the values of quantities, in place of quantities themselves, and various other such contrivances may be used, which will render the solution much less complicated. (See Ex. 3. 7. and 9.)

CHAP.

C H A P. III.

P A R T II.

General Solution of Problems.

IN the solutions of the questions in the preceding part, the given quantities (being numbers) disappear in the last conclusion, so that no general rules for like cases can be deduced from them. But if letters are used to denote the known quantities, as well as the unknown, a general solution may be obtained, because, during the whole course of the operation, they retain their original form. Hence also the connection of the quantities will appear in such a manner as to discover the necessary limitations of the data, when there are any, which is essential to the perfect solution of

a problem. From this method, too, it is easy to derive a synthetical demonstration of the solution.

When letters, or any other such symbols, are employed to express all the quantities, the algebra is sometimes called *specious* or *literal*.

E X A M P L E VIII.

To find two numbers, of which the sum and difference are given.

Let s be the given sum, and d the given difference. Also, let x and y be the two numbers sought.

$$\begin{aligned} x + y &= s \\ x - y &= d \\ \text{Whence } \begin{cases} x = s - y \\ x = d + y \end{cases} \\ d + y &= s - y \\ 2y &= s - d \end{aligned}$$

$$y = \frac{s - d}{2}$$

$$\text{And } x = \frac{s + d}{2}$$

Thus, let the given sum be 100, and the difference 24.

$$\text{Then } x = \left(\frac{s + d}{2} = \frac{124}{2} = \right) 62 \text{ and } y = \left(\frac{s - d}{2} = \frac{76}{2} = \right) 38.$$

In the same manner may the canon be applied to any other values of s and d . By reversing the steps in the operation, it is easy to shew, that if $x = \frac{s+d}{2}$ and $y = \frac{s-d}{2}$, the sum of x and y must be s , and their difference d .

EXAMPLE IX.

If A and B together can perform a piece of work in the time a, A and C together in the time b, and B and C together in the time c, in what time will each of them perform it alone?

Let A perform the work in the time x , B in y , and C in z ; then as the work is the same in all cases, it may be represented by unity.

K

By

	1	$(x:1::a:) \frac{a}{x} =$	} to the work performed by	A in a days
	2	$(y:1::a:) \frac{a}{y} =$		B in a days
	3	$(x:1::b:) \frac{b}{x} =$		A in b days
	4	$(z:1::b:) \frac{b}{z} =$		C in b days
By the question	5	$(y:1::c:) \frac{c}{y} =$		B in c days
	6	$(z:1::c:) \frac{c}{z} =$		C in c days
	7	$\frac{a}{x} + \frac{a}{y} = 1$ and $ay + ax = xy$		
	8	$\frac{b}{x} + \frac{b}{z} = 1$ and $bz + bx = xz$		
	9	$\frac{c}{y} + \frac{c}{z} = 1$ and $cz + cy = zy$		
Mult. 7th by $\frac{bc}{xy}$	10	$\frac{abc}{x} + \frac{abc}{y} = bc$		
Mult. 8th by $\frac{ac}{xz}$	11	$\frac{abc}{x} + \frac{abc}{z} = ac$		
Mult. 9th by $\frac{ab}{zy}$	12	$\frac{abc}{y} + \frac{abc}{z} = ab$		
Add 10th, 11th, 12th	13	$\frac{2abc}{x} + \frac{2abc}{y} + \frac{2abc}{z} = bc + ac + ab$		
From 13th subtr. twice 10th	14	$\frac{2abc}{z} = ac + ab - bc$ & $z = \frac{2abc}{ac+ab-bc}$		
From 13th subtr. twice 11th	15	$\frac{2abc}{y} = bc + ab - ac$ & $y = \frac{2abc}{bc+ab-ac}$		
From 13th subtr. twice 12th	16	$\frac{2abc}{x} = bc + ac - ab$ & $x = \frac{2abc}{bc+ac-ab}$		

Example in numbers. Let $a=8$ days,
 $b=9$ days, and $c=10$; then $x=14\frac{34}{49}$
 $y=17\frac{23}{41}$, and $z=23\frac{7}{31}$. It appears like-

likewise that a , b , c , must be such, that the product of any two of them must be less than the sum of these two multiplied by the third. This is necessary to give positive values of x , y , and z , which alone can take place in this question. Besides, if x , y , and z be assumed as any known numbers whatever, and if values of a , b , and c be deduced from steps 7th, 8th, and 9th, of the preceding operation, it will appear, that a , b , and c will have the property required in the limitation here mentioned.

If a , b , and c were such, that any of the quantities, x , y or z , became equal to 0, it implies, that one of the agents did nothing in the work. If the values of any of these quantities be negative, the only supposition which could give them any meaning, would be, that some of the agents, instead of promoting the work, either obstructed it, or undid it to a certain extent.

EXAMPLE X.

In question 5th, let the first courier travel p miles in q hours; the second r miles in s hours; let the interval between their setting out be a .

Then by working as formerly.

$$x = \frac{qra}{qr - ps}.$$

If particular values be inserted for these letters, a particular solution will be obtained for that case. Let them denote the numbers in Example 5.

$$\text{Then } x = \left(\frac{qra}{qr - ps} = \frac{5 \times 5 \times 8}{5 \times 5 - 7 \times 3} = \frac{200}{4} = \right) 50.$$

Here it is obvious, that qr must be greater than ps , else the problem is impossible; for then the value of x would either be infinite or negative. This limitation appears also from the nature of the question, as the second courier must travel at a greater rate than the first, in order to overtake him. For the rate of the first courier is

is to the rate of the second as $\frac{p}{q}$ to $\frac{r}{s}$, that is, as ps to qr ; and therefore qr must be greater than ps .

SCHOLIUM.

Sometimes when there are many known quantities in a general solution, it may simplify the operation, to express certain combinations of them by new letters, still to be considered as known.

CHAP.

is to the rate of the second as $\frac{1}{2}$ to $\frac{1}{3}$, that is
as 31 to 27; and therefore q will be great-
er than p .

SCHOLIUM.

Sometimes when there are many known
quantities in a general solution, it may sim-
plify the operation, to express certain com-
binations of them by new letters, till to be
considered as known.

CHAP.

C H A P. IV.

Of Involution and Evolution.

IN order to resolve equations of the higher orders, it is necessary to premise the rules of INVOLUTION and EVOLUTION.

L E M M A.

The reciprocals of the powers of a quantity may be expressed by that quantity, with negative exponents of the same denomination. That is, the series a , 1,

$\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^m}$, &c. may be expressed by a^1 , a^0 , a^{-1} , a^{-2} , a^{-3} , a^{-m} , &c.

For

For the rule for dividing the powers of the same root was to subtract the exponents; if then the index of the divisor be greater than that of the dividend, the index of the quotient must be negative.

Thus, $\frac{a^2}{a^3} = a^2 - 3 = a^{-1}$. Also, $\frac{a^3}{a^3} = \frac{1}{a^1}$.

$\frac{a^m}{a^m} = a^m - m = a^0$. And, $\frac{a^m}{a^m} = 1$. and so on of others.

Cor. 1. Hence any quantity which multiplies either the numerator or denominator of a fraction, may be transposed from the one to the other, by changing the sign of its index.

Thus, $\frac{x}{y^2} = xy^{-2}$. And $\frac{a^2x}{y^3} = \frac{a^2}{y^3x^{-1}}$, &c.

Cor. 2. From this notation, it is evident that these *negative powers*, as they are called, are multiplied by adding, and divided by subtracting their exponents.

Thus,

Thus, $a^{-2} \times a^{-3} = a^{-5}$.

Or, $\frac{1}{a^2} \times \frac{1}{a^3} = \frac{1}{a^5} = a^{-5}$.

$\frac{a^{-1}}{a^{-3}} = a^2$ Or, $\frac{1}{a^3} \cdot \frac{1}{a} \left(\frac{a^3}{a} \right) = a^2$.

I. Of Involution.

To find any power of any quantity, is the business of involution.

Case I. When the quantity is simple.

Rule. Multiply the exponents of the letters by the index of the power required, and raise the coefficient to the same power.

Thus, the 2d power of a , is $a^1 \times 2 = a^2$

The 3d power of $2a^2$, is $8a^2 \times 3 = 8a^6$

The 3d power of $3ab^3$, is $27a^1 \times 3b^3 \times 3 = 27a^3b^9$.

For the multiplication would be performed by the continued addition of the exponents; and this multiplication of them is equivalent. The same rule holds also when the signs of the exponents are negative.

L

Rule

Rule for the Signs.

If the sign of the given quantity is +, all its powers must be positive. If the sign is —, then all its powers whose exponents are even numbers, are positive; and all its powers whose exponents are odd numbers, are negative.

This is obvious from the rule for the signs in multiplication.

The last part of it implies the most extensive use of the signs + and —, by supposing that a negative quantity may exist by itself.

Case II. When the quantity is compound.

Rule. The powers must be found by a continual multiplication of it by itself.

Thus, the square of $x + \frac{a}{2}$ is found by multiplying it into itself. The product is $x^2 + ax + \frac{a^2}{4}$. The cube of $x + \frac{a}{2}$ is got by multiplying

multiplying the square already found by the root, &c.

Fractions are raised to any power, by raising both numerator and denominator to that power, as is evident from the rule for multiplying fractions, in chap. I. p. 2.

The involution of compound quantities is rendered much easier by the binomial theorem; for which, see chap. VI.

Note. The square of a binomial consists of the squares of the two parts, and twice the product of the two parts.

II. *Of Evolution.*

Evolution is the reverse of involution, and by it powers are resolved into their roots.

Def. The root of any quantity is expressed by placing before it $\sqrt{\quad}$ (called a *radical sign*) with a small figure above it, denoting the denomination of that root.

Thus,

Thus, the square root of a , is $\sqrt[2]{a}$ or $\sqrt{a}$

The cube root of bc , is $\sqrt[3]{bc}$

The 4th root of $a^2b - x^3$, is $\sqrt[4]{a^2b - x^3}$

The m th root of $c^2 - dx$, is $\sqrt[m]{c^2 - dx}$

General Rule for the Signs.

1. *The root of any positive power may be either positive or negative, if it is denominated by an even number ; if the root is denominated by an odd number, it is positive only.*
2. *If the power is negative, the root also is negative, when it is denominated by an odd number.*
3. *If the power is negative, and the denomination of the root even, then no root can be assigned.*

This rule is easily deduced from that given in involution, and supposes the same extensive use of the signs $+$ and $-$. If it is applied to abstract quantities in which a contrariety cannot be supposed, any root of a positive quantity must be positive only, and

and any root of a negative quantity, like itself, is unintelligible.

In the last case, though no root can be assigned, yet sometimes it is convenient to set the radical sign before the negative quantity, and then it is called an *impossible* or *imaginary* root.

The root of a positive power, denominated by an even number, has often the sign $\pm$ before it, denoting that it may have either $+$ or $-$.

The radical sign may be employed to express any root of any quantity whatever; but sometimes the root may be accurately found by the following rules, and when it cannot, it may often be more conveniently expressed by the methods now to be explained.

Case I. When the quantity is simple.

Rule. Divide the exponents of the letters by the index of the root required, and prefix the root of the numeral coefficient.

1. The exponents of the letters may be multiples of the index of the root, and the root of the coefficient may be extracted.

Thus,

Thus, the square root of $a^4 = a^{\frac{4}{2}} = \pm a^2$

$$\sqrt[3]{27a^6} = 3a^{\frac{6}{3}} = 3a^2$$

$$\sqrt[4]{a^4b^{12}} = a^{\frac{4}{4}}b^{\frac{12}{4}} = \pm ab^3.$$

2. The exponents of the letters may not be multiples of the index of the root, and then they become fractions; and when the root of the coefficient cannot be extracted, it may also be expressed by a fractional exponent, its original index being understood to be 1.

$$\text{Thus, } \sqrt{16a^3b^2} = 4a^{\frac{3}{2}}b$$

$$\sqrt[3]{7ax^3} = 7^{\frac{1}{3}}a^{\frac{1}{3}}x = \sqrt[3]{7} \times a^{\frac{1}{3}}x.$$

As evolution is the reverse of involution, the reason of the rule is evident.

The root of any fraction is found by extracting that root out of both numerator and denominator.

Case II. When the quantity is compound.

1. To extract the square root.

RULE,

R U L E.

1. *The given quantity is to be ranged according to the powers of the letters, as in division.*

Thus, in the example $a^2 + 2ab + b^2$, the quantities are ranged in this manner.

2. *The square root is to be extracted out of the first term (by preceding rules), which gives the first part of the root sought. Subtract its square from the given quantity, and divide the first term of the remainder by double the part already found, and the quotient is the second term of the root.*

Thus, in this example, the remainder is $2ab + b^2$; and $2ab$ being divided by $2a$, the double of the part found, gives $+b$ for the second part of the root.

3. *Add this second part to double of the first, and multiply their sum by the second part: Subtract the product from the last remainder,*

der, and if nothing remain, the square root is obtained. But, if there is a remainder, it must be divided by the double of the parts already found, and the quotient would give the third part of the root; and so on.

In the last example, it is obvious, that $a+b$ is the square root sought.

The entire operation is as follows:

$$\begin{array}{r}
 a^2 + 2ab + b^2 \quad (a+b) \\
 \underline{a^2} \\
 2a+b \quad + 2ab + b^2 \\
 \times b \quad \underline{2ab + b^2} \\
 *
 \end{array}$$

$$\begin{array}{r}
 x^4 - ax^2 + \frac{a^2}{4} \quad (x^2 - \frac{a}{2}) \\
 \underline{x^4} \\
 2x^2 - \frac{a}{2} \quad - ax^2 + \frac{a^2}{4} \\
 \times -\frac{a}{2} \quad \underline{- ax^2 + \frac{a^2}{4}} \\
 *
 \end{array}$$

The

The reason of this rule appears from the composition of a square.

2. To extract any other root.

Rule. Range the quantity according to the dimensions of its letters, and extract the said root out of the first term, and that shall be the first member of the root required. Then raise this root to a dimension lower by unit, than the number that denominates the root required, and multiply the power that arises, by that number itself: divide the second term of the given quantity by the product, and the quotient shall give the second member of the root required. In like manner are the other parts to be found, by considering those already got as making one term.

Thus, the fifth root of

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5(a+b)$$

$$5a^4 \quad) \quad 5a^4b$$

M

And

And $a+b$ raised to the 5th power is the given quantity, and therefore it is the root sought.

In evolution it will often happen, that the operation will not terminate, and the root will be expressed by a series.

Thus, the square root of a^2+x^2 becomes a series.

$$\begin{array}{r}
 a^2+x^2 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}, \&c. \right. \\
 \hline
 2a + \frac{x^2}{2a} \left. \right) * + x^2 \\
 \times \frac{x^2}{2a} \qquad \qquad = x^2 + \frac{x^4}{4a^2} \\
 \hline
 2a + \frac{x^2}{a} - \frac{x^4}{8a^3} \left. \right) * - \frac{x^4}{4a^2} \\
 \times - \frac{x^4}{8a^3} \qquad \qquad = - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
 \hline
 * + \frac{x^6}{8a^4} - \frac{x^8}{64a^6} \\
 \&c.
 \end{array}$$

The extraction of roots by series is much facilitated by the binomial theorem (Chap. VI. Sect. 3.). By similar rules, founded on the same principles, are the roots of numbers to be extracted.

III. Of Surds.

Def. Quantities with fractional exponents are called *Surds* or *Imperfect Powers*.

Such quantities are also called *irrational*, in opposition to others with integral exponents, which are called *rational*.

Surds may be expressed either by the fractional exponents, or by the radical sign, the denominator of the fraction being its index ; and hence the orders of surds are denominated from this index.

In the following operations, however, it is generally convenient to use the notation by the fractional exponents.

$$a^{\frac{1}{2}} = \sqrt[3]{a} \cdot \sqrt{4ab^2} = 2ba^{\frac{1}{2}}. \sqrt[4]{a^3b^2} = a^{\frac{3}{4}}b^{\frac{1}{2}}.$$

The

The operations concerning surds depend on the following principle. *If the numerator and denominator of a fractional exponent be both multiplied or both divided by the same quantity, the value of the power is the same.* Thus $a^{\frac{m}{n}} = a^{\frac{mc}{nc}}$; for let $a^{\frac{m}{n}} = b$; then $a^m = b^n$, and $a^{mc} = b^{nc}$, and extracting the root nc , $a^{\frac{mc}{nc}} = b^{\frac{nc}{nc}} = b = a^{\frac{m}{n}}$.

Lem. A rational quantity may be put into the form of a surd, by reducing its index to the form of a fraction of the same value.

$$\text{Thus } a = a^{\frac{1}{1}} = \sqrt[1]{a^1}$$

$$a^3 b = a^{\frac{6}{2}} b^{\frac{3}{2}} = \sqrt[2]{a^6 b^3}$$

Prob. I. To reduce surds of different denominations to others of the same value, and of the same denomination.

Rule. Reduce the fractional exponents to others of the same value and having the same common denominator.

Ex.

Ex. $\sqrt{a}$, $\sqrt[3]{b^2}$ or $a^{\frac{1}{2}}$, $b^{\frac{2}{3}}$

but $a^{\frac{1}{2}} = a^{\frac{2}{4}}$ and $b^{\frac{2}{3}} = b^{\frac{4}{6}}$.

therefore $\sqrt{a}$, and $\sqrt[3]{b^2}$ are respectively equal to $\sqrt[6]{a^3}$ and $\sqrt[6]{b^4}$.

Prob. II. To multiply and divide surds.

1. *When they are surds of the same rational quantity, add and subtract their exponents.*

Thus, $a^{\frac{1}{2}} \times a^{\frac{1}{4}} = a^{\frac{1}{2} + \frac{1}{4}} = a^{\frac{3}{4}} = \sqrt[4]{a^3}$

$\sqrt{a^2 - b^2} \times \sqrt{a^2 - b^2} = (a^2 - b^2)^{\frac{1}{2}} = (a^2 - b^2)^{\frac{2}{4}} = \sqrt[4]{(a^2 - b^2)^2} = \sqrt[4]{a^4 - 2a^2b^2 + b^4}$

$\sqrt[3]{a^2 - b^2} \times \sqrt[3]{a^2 - b^2} = (a^2 - b^2)^{\frac{1}{3}} = (a^2 - b^2)^{\frac{2}{6}} = \sqrt[6]{(a^2 - b^2)^2} = \sqrt[6]{a^4 - 2a^2b^2 + b^4}$

2. *If they are surds of different rational quantities, let them be brought to others of the same denomination, if already they are not, by prob. 1. Then, by multiplying or dividing these rational quantities, their product or quotient may be set under the common radical sign.*

Thus $\sqrt[m]{a} \times \sqrt[n]{b} = a^{\frac{1}{m}} b^{\frac{1}{n}} = \sqrt[mn]{a^n b^m}$

$\frac{\sqrt{a^2 - b^2}}{\sqrt{a + b}} = \sqrt{\frac{a^2 - b^2}{a + b}} = \sqrt{a - b}$

$$\begin{aligned} \frac{\sqrt[4]{a^3b^2}}{\sqrt[3]{a^2b}} &= \frac{a^{\frac{3}{4}}b^{\frac{2}{4}}}{a^{\frac{2}{3}}b^{\frac{1}{3}}} = a^{\frac{3}{4}-\frac{2}{3}}b^{\frac{2}{4}-\frac{1}{3}} = a^{\frac{1}{12}}b^{\frac{1}{12}} \\ &= \sqrt[12]{ab} \end{aligned}$$

If the surds have any rational coefficients, their product or quotient must be prefixed. Thus, $a\sqrt{m} \times b\sqrt{n} = ab\sqrt{mn}$. It is often convenient, in the operations of this problem, not to bring the surds of simple quantities to the same denomination, but to express their product or quotient without the radical sign, in the same manner as if they were rational quantities. Thus, the product in Ex. 1. may be $a^{\frac{1}{2}}b^{\frac{1}{2}}$, and the quotient in Ex. 3. $a^{\frac{1}{12}}b^{\frac{1}{12}}$.

Cor. If a rational coefficient be prefixed to a radical sign, it may be reduced to the form of a surd by the lemma, and multiplied by this problem; and conversely, if the quantity under the radical sign be divisible by a perfect power of the same denomination,

nomination, it may be taken out, and its root prefixed as a coefficient.

$$a\sqrt{b} = \sqrt{a^2b}; 2 \times \sqrt[3]{a} = \sqrt[3]{8a}.$$

$$\text{Conv. } \sqrt{a^2b^3} = ab\sqrt{b}; \sqrt{4a^2 - 8a^2b} = 2a\sqrt{1 - 2b}.$$

Even when the quantity under the radical sign is not divisible by a perfect power, it may be useful sometimes to divide surds into their component factors, by reversing the operation of this problem.

$$\text{Thus } \sqrt{ab} = \sqrt{a} \times \sqrt{b}, \sqrt[3]{a^2b - bx^2} = \sqrt[3]{ba - bx^2}$$

$$\times \sqrt[3]{a+x} = b \times \sqrt[3]{a-x} \times \sqrt[3]{a+x}.$$

PROB. III. To involve or evolve surds.

This is performed by the same rules as in other quantities, by multiplying or dividing their exponents by the index of the power or root required.

The notation by negative exponents, mentioned in the lemma at the beginning of this chapter, is applicable to fractional exponents, in the same manner as to integers.

SCHO-

S C H O L I U M.

The application of the rules of this chapter to the resolving of equations, shall be explained in the succeeding chapters, which treat of the solution of the different classes of them ; but some examples of their use in preparing equations for a solution, are the following.

If a member of an equation be a surd root, then the equation may be freed from any surd, by bringing that member first to stand alone upon one side of the equation, and then taking away the radical sign from it, and raising the other side to the power denominated by the index of that surd.

This operation becomes a necessary step towards the solution of an equation, when any of the unknown quantities are under the radical sign.

Example, If $3\sqrt{x^3 - a^3} + 2y = a + y$

Then $3\sqrt{x^3 - a^3} = a - y$

and $9 \times x^3 - a^3 = a^2 - 2ay + y^2$

If

If the unknown quantity be found only under the radical sign, and only of the first dimension, the equation will become simple, and may be resolved by the preceeding rules.

Thus, if $\sqrt[3]{4x+16}+5=9$

Then $\sqrt[3]{4x+16}=4$

And $4x+16=64$

$4x=48$

And, $x=12$

If $\sqrt[m]{a^2x-b^2x}=a$

Then, $a^2x-b^2x=a^m$

$$x=\frac{a^m}{a^2-b^2}$$

If the unknown quantity in a final equation has fractional exponents, by means of the preceding rules a new equation may be substituted, in which the exponents of the unknown quantity are integers.

Thus, if $x^{\frac{1}{2}}+3x^{\frac{3}{2}}=10$, by reducing the surds to the same denomination, it becomes $x^{\frac{1}{6}}+3x^{\frac{4}{6}}=10$; and if $z=x^{\frac{1}{6}}$ then $z^3+3z^4=10$, and if this equation be resol-

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ved

ved from a value of z a value of x may be got by the rules of the next chapter. Thus also, if $x + 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} = 100$. If $x^{\frac{1}{6}} = z$, this equation becomes $z^6 + 2z^3 - 3z^2 = 100$.

In general, if $x^{\frac{p}{q}} + x^{\frac{m}{n}} = a$. by reducing the surds to the same denomination $x^{\frac{pn}{qn}} + x^{\frac{qm}{qn}} = a$, and if $x^{\frac{1}{qn}} = z$, then the equation is $z^{pn} + z^{qm} = a$ in which the exponents of z are integers; and z being found, x is to be found from the equation $x^{\frac{1}{qn}} = z$.

CHAP.

C H A P. V.

EQuations were divided into orders according to the highest index of the unknown quantity in any term. (Chap. III.)

Equations are either *pure* or *adfectèd*.

Def. 1. A *pure equation* is that in which only one power of the unknown quantity is found.

2. An *adfectèd equation*, is that in which different powers of the unknown quantity are found in the several terms.

Thus, $a^2 + ax^2 = b^3$, $ax^2 - b^2 = m^2 + x^2$ are pure equations.

And $x^2 - ax = b^2$, $x^3 + x^2 = 17$, are adfectèd.

I. *Solution of pure Equations.*

Rule. Make the power of the unknown quantity to stand alone by the rules formerly given, and then extract the root of the same denomination out of both sides, which will give the value of the unknown quantity.

E X A M P L E S.

$$\begin{array}{ll}
 \text{If } a^3 + ax^3 = b^3 & ax^m - b = x^m - c \\
 ax^3 = b^3 - a^3 & ax^m - x^m = b - c \\
 x^3 = \frac{b^3 - a^3}{a} & x^m = \frac{b - c}{a - 1} \\
 x = \sqrt[3]{\frac{b^3 - a^3}{a}} & x = \sqrt[m]{\frac{b - c}{a - 1}}
 \end{array}$$

The index of the power may also be fractional; as in the last example m may be any number whatever. Let $m = \frac{1}{2}$, then as before,

$$x^m = x^{\frac{1}{2}} = \frac{b-c}{a-1}$$

$$\text{And } x = \frac{\overline{b-c}}{a-1} = \frac{b^2-2bc+c^2}{a^2-2a+1}.$$

Sometimes different powers of the unknown quantity are found in the equation, yet the several terms may form on one side a perfect power, of which the root being extracted, the equation will become simple.

Thus, if $x^3 - 12x^2 + 48x = 98$. It is easy to observe that $x^3 - 12x^2 + 48x - 64 = 34$; forming a complete cube, of which the root being extracted, $x - 4 = \sqrt[3]{34}$. And $x = 4 + \sqrt[3]{34}$.

EXAMPLE I.

To find four continued proportionals, of which the sum of the extremes is 56, and the sum of the means 24.

To resolve the question in general terms, let the sum of the extremes be a , the sum of the means b , and let the difference of the extremes be called z , and the difference of the means y . Then by Ex. 8. Chap. 3.

The

The proportionals are	1	$\frac{a+z}{2} : \frac{b+y}{2} : \frac{b-y}{2} : \frac{a-z}{2}$
Mult. by 2 and still	2	$a+z : b+y : b-y : a-z$
From the three first	3	$ab - ay + bz - zy = b^2 + 2by + y^2$
From the three last	4	$ab + ay - bz - zy = b^2 - 2by + y^2$
3d added to 4th	5	$2ab - 2zy = 2b^2 + 2y^2$
4th subtr. from 3d	6	$2bz - 2ay = 4by$
6th reduced	7	$z = \frac{2by + ay}{b}$
7th subtr. for z in 5th	8	$2ab - 2 \times \frac{2by^2 + ay^2}{b} = 2b^2 + 2y^2$
Transp. and divide 8th by $\frac{2}{b}$	9	$ab^2 - b^3 = 3by^2 + ay^2$
	10	$\frac{ab^2 - b^3}{3b + a} = y^2$ and $y = \sqrt{\frac{ab^2 - b^3}{3b + a}}$
In numbers	11	$y = \sqrt{\frac{ab^2 - b^3}{3b + a}} = b \sqrt{\frac{a-b}{3b+a}} = 24 \sqrt{\frac{32}{128}} = 12$
	12	$z = \frac{2b+a}{b} \times y = 52$

Hence the four proportionals are 54, 18, 6, 2; and it appears that b must not be greater than a , otherwise the root becomes impossible, and the problem would also be impossible; which limitation might be deduced also from prop. 25. V. of Euclid.

II. *Solution of adfected Quadratic Equations.*

Adfected equations of different orders are resolved by different rules, successively to be explained.

An adfected quadratic equation, (commonly called a quadratic) involves the unknown quantity itself, and also its square : It may be resolved by the following

R U L E.

1. *Transpose all the terms involving the unknown quantity, to one side, and the known terms to the other ; and so that the term containing the square of the unknown quantity may be positive.*
2. *If the square of the unknown quantity is multiplied by any coefficient, all the terms of the equation are to be divided by it, so that the coefficient of the square of the unknown quantity may be 1.*
3. *Add to both sides the square of half the coefficient of the unknown quantity itself, and*

and the side of the equation involving the unknown quantity will be a complete square.

4. Extract the square root from both sides of the equation, by which it becomes simple, and by transposing the above mentioned half coefficient, a value of the unknown quantity is obtained in known terms, and therefore the equation is resolved.

The reason of this rule is manifest from the composition of the square of a binomial, for it consists of the squares of the two parts, and twice the product of the two parts. (*Note*, at the end of Chap. IV.)

The different forms of quadratic equations, expressed in general terms, being reduced by the first and second parts of the rule, are these ;

1. $x^2 + ax = b^2$

2. $x^2 - ax = b^2$

3. $x^2 - ax = -b^2$

Case 1. $x^2 + ax = b^2$

$$x^2 + ax + \frac{a^2}{4} = b^2 + \frac{a^2}{4}$$

$$x + \frac{a}{2} = \pm \sqrt{b^2 + \frac{a^2}{4}}$$

$$x = \pm \sqrt{b^2 + \frac{a^2}{4}} - \frac{a}{2}$$

Case 2. $x^2 - ax = b^2$

$$x^2 - ax + \frac{a^2}{4} = b^2 + \frac{a^2}{4}$$

$$x - \frac{a}{2} = \pm \sqrt{b^2 + \frac{a^2}{4}}$$

$$x = \frac{a}{2} \pm \sqrt{b^2 + \frac{a^2}{4}}$$

Case 3. $x^2 - ax = -b^2$

$$x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - b^2$$

$$x - \frac{a}{2} = \pm \sqrt{\frac{a^2}{4} - b^2}$$

$$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2}$$

Of these cases it may be observed,

1. That if it be supposed, that the square root of a positive quantity may be either positive or negative, according to the most extensive use of the signs, every quadratic

O

equation

equation will have two roots, except such of the third form, whose roots become impossible.

2. It is obvious, that, in the two first forms, one of the roots must be positive, and the other negative.

3. In the third form, if $\frac{a^2}{4}$ or the square of half the coefficient of the unknown quantity be greater than b^2 , the known quantity, the two roots will be positive. If $\frac{a^2}{4}$ be equal to b^2 , the two roots then become equal.

But if in this third case $\frac{a^2}{4}$ is less than b^2 , the quantity under the radical sign becomes negative, and the two roots are therefore impossible. This may be easily shewn to arise from an impossible supposition in the original equation.

4. If the equation, however, express the relation of magnitudes abstractly considered, where a contrariety cannot be supposed to take place ; the negative roots cannot be of use, or rather there are no such roots ; for then a negative quantity by itself is unintelligible,

intelligible, and therefore the square root of a positive quantity must be positive only. Hence, in the two first cases, there will be only one root; but in the third, there will be two. For in this third case, $x^2 - ax = -b^2$, or, $ax - x^2 = b^2$, it is obvious, that x may be either greater or less than $\frac{1}{2}a$, and yet $a - x$ may be positive; and hence $a - x \times x = ax - x^2$ may also be positive, and may be equal to a given positive quantity b^2 : therefore the square root of $x^2 - ax + \frac{1}{4}a^2$ may be either $x - \frac{1}{2}a$ or $\frac{1}{2}a - x$, and both these quantities also positive.

Let then $x - \frac{a}{2} = \sqrt{\frac{a^2}{4} - b^2}$ and $x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b^2}$. Also let $\frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b^2}$ and hence $x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b^2}$. and these are the same two positive roots as were obtained by the general rule.

The general rule is usually employed, even in questions where negative numbers cannot take place, and then the negative roots of the two first forms are neglected.

Sometimes

Sometimes even only one of the positive roots of the third case can be used, and the other may be excluded by a particular condition in the question. When an impossible root arises in the solution of a question, and if it be resolved in general terms, the necessary limitation of the data will be discovered.

When a question can be so stated, as to produce a pure equation, it is generally to be preferred to an affected. Thus the question in the preceding section, by the most obvious notation, would produce an affected equation.

II. *Solution of Questions producing Quadratic Equations.*

The expression of the conditions of the question by equations, or the stating of it, and the reduction likewise of these equations, till we arrive at a quadratic equation, involving only one unknown quantity and its square, are effected by the same rules which were given for the solution of simple equations, in Chap. III.

E X-

E X A M P L E II.

*One lays out a certain sum of money in goods,
which he sold again for L. 24, and gain-
ed as much per cent. as the goods cost him :
I demand what they cost him.*

If the money laid out be	1	y
The gain will be	2	$24 - y$
But this gain is	}	$\frac{2400 - 100y}{y}$ per cent.
$(y:24-y::100:)$		
Therefore by question	4	$y = \frac{2400 - 100y}{y}$
And by mult, and tr.	5	$y^2 + 100y = 2400$
Completing the square	6	$y^2 + 100y + 50^2 = 2400 + 2500 = 4900$
Extr. the root	7	$y + 50 = \pm \sqrt{4900} = 70$
Transp.	8	$y = \pm 70 - 50 = 20 \text{ or } -120.$

The answer is 20 l. which succeeds. The other root, —120, has no place in this example, a negative number being here unintelligible.

Any quadratic equation may be resolved also by the general canons at the beginning of this section. That arising from this question, (No. 5.) belongs to case I. and $a=100$, $b^2=2400$; therefore,

y

$$y = \left(-\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2} \right) - \frac{100}{2} \pm \sqrt{\frac{100^2}{4} + 2400} = 20 \text{ or } -120 \text{ as before.}$$

E X A M P L E III.

What two numbers are those, whose difference is 15, and half of whose product is equal to the cube of the lesser?

Let the lesser number be

The greater is

By question

Divide by x and mult. by 2.

4th prepared

Complete square

Ext. $\sqrt{}$.

transp.

$$\begin{array}{r|l}
 1 & x \\
 2 & x + 15 \\
 3 & x^2 + 15x \\
 & \underline{ - x^3} \\
 & 2 = x^3 \\
 4 & x + 15 = 2x^2 \\
 5 & x^2 - \frac{x}{2} = \frac{15}{2} \\
 6 & x^2 - \frac{x}{2} + \frac{1}{16} = \frac{15}{2} + \frac{1}{16} = \frac{121}{16} \\
 7 & x - \frac{1}{4} = \pm \frac{11}{4} \\
 8 & x = 3 \text{ or } -\frac{5}{2}
 \end{array}$$

The numbers therefore are 3 and 18, which answer the conditions. This is an example of case 2d, and the negative root is neglected.

And

((III))

A solution, indeed, may be represented by means of the negative root $-\frac{5}{2}$; for then the other number is $(x + 15) - \frac{5}{2} + 15 = \frac{25}{2}$. And $\frac{1}{2} \times \frac{25}{2} \times -\frac{5}{2}$, is equal to the cube of $-\frac{5}{2}$. Such a solution, though useless, and even absurd, it is plain must correspond to the conditions, if those rules with regard to the signs be used in the application of it, by which it was itself deduced. The same observation may be extended even to impossible roots, which being assumed as the answer of a question, must, by reversing the steps of the investigation, correspond to the original equations, by which the conditions of that question were expressed.

E X A M P L E IV.

To find two numbers whose sum is 100, and whose product is 2059.

Let

Let the given sum $100=a$, the product $2059=b$, and let one of the numbers sought be x , the other will be $a-x$. Their product is $ax-x^2$.

Therefore by question 1 $ax-x^2=b$ or $x^2-ax=-b$

Complete the square 2 $x^2-ax+\frac{a^2}{4}=\frac{a^2}{4}-b$

Ext. $\sqrt{\quad}$ 3 $x-\frac{a}{2}=\pm\sqrt{\frac{a^2}{4}-b}$

Transp. 4 $x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b}$

And the other number 5 $a-x=\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-b}$

By inserting numbers, $x=71$ or 29 and $a-x=29$ or 71 , so that the two numbers sought are 71 and 29 .

Here it is to be observed, that b must not be greater than $\frac{a^2}{4}$, else the roots of the equation would be impossible; that is, the given product must not be greater than the square of half the given sum of the numbers sought. This limitation can easily be shewn from other principles; for, the greatest possible product of two parts, into which any

any number may be divided, is when each of them is a half of it. If b be equal to $\frac{a^2}{4}$, there is only one solution, and $x = \frac{a}{2}$, also $a - x = \frac{a}{2}$.

EXAMPLE V.

There are three numbers in continual geometrical proportion: The sum of the first and second is 10, and the difference of the second and third is 24. What are the numbers?

Let the first be	1 z
The second will be	2 $10 - z$
And the third	3 $34 - z$
Since $z : 10 - z : 34 - z$	4 $z^2 - 20z + 100 = 34z - z^2$
Transp.	5 $2z^2 - 54z = -100$
Divid.	6 $z^2 - 27z = -50$
Compl. the square	7 $\left[z - 27z + \frac{27^2}{2} \right]^2 = \frac{729}{4} - 50 = \frac{529}{4}$
Extract the $\sqrt{}$	8 $z - \frac{27}{2} = \pm \sqrt{\frac{529}{4}} = \pm \frac{23}{2}$
Transp.	9 $z = \frac{27}{2} \pm \frac{23}{2} = 25 \text{ or } 2.$

But though there are two positive roots in this equation, yet one of them only can

P

here

here be of use, the other being excluded by a condition in the question. For, as the sum of the first and second is 10, 25 cannot be one of them : 2 therefore is the first, and the proportionals will be 2, 8, 32.

This restriction will also appear from the explanation given of the third form, to which this equation belongs. For z may be less than $\frac{27}{2}$, but from the first condition of the question it cannot be greater ; hence the quantity $z^2 - 27z + \frac{27^2}{4}$ can have only one square root, viz. $\frac{27}{2} - z$; and this being put equal to $\sqrt{\frac{529}{4}}$, we have by transposition $z = \frac{27}{2} - \frac{23}{2} = 2$ which gives the only just solution of the question.

From the other root, indeed, a solution of the question may be represented by means of a negative quantity. If the first then be 25, the three proportionals will be 25, -15, 9. These also must answer the conditions, according to the rules given for negative

gative quantities, though such a solution has no proper meaning.

Besides, it is to be observed, that, if the following question be proposed, ‘ *to find three numbers in geometrical proportion, so that the difference of the 1st and 2d may be 10, and the sum of the 2d and 3d may be 24.*’ The equation in step 6th will be produced; for, if the 1st be z , the 2d is $z - 10$, and the 3d $34 - z$, and therefore $34z - z^2 = z^2 - 20z + 100$, the very same equation as in step 4th. In this question it is plain that the root 25 only can be useful, and the three proportionals are 25, 15, 9.

But the necessary limitations of such a problem are properly to be derived from a general notation. Let the sum of the two first proportionals be a , and the difference of the two last b . If a is not greater than b , the first term must be the least; but, if a be greater than b , the first term may be either the greatest or the least.

When the first term is the least, the proper notation of the three terms is z , $a - z$,

a

$a+b-z$, and the equation when ordered is $z^2 - \frac{3a+b}{2}z = -\frac{a^2}{2}$. If the first term be the greatest, and then a is greater than b , the notation of the terms is z , $a-z$, $a-b-z$, and the corresponding equation is $z^2 - \frac{3a-b}{2}z = -\frac{a^2}{2}$.

Of the first of these equations it may be observed, that, whatever be the value of a and b , the square of $\frac{3a+b}{4}$, viz. of half the coefficient of z , is greater than $\frac{a^2}{2}$, and therefore the roots are always possible. If the square be completed, and the roots extracted, they become $z - \frac{3a+b}{4} = \frac{\sqrt{3a+b)^2 - 8a^2}}{4}$, and $\frac{3a+b}{4} - z = \frac{\sqrt{3a+b)^2 - 8a^2}}{4}$. But in this case z is the least of the three terms, and therefore a is greater than $2z$, or $\frac{a}{2}$ is greater than z ; much more then is $\frac{3a+b}{4}$ greater than z , and therefore the second root only can be admitted,

mitted, and $z = \frac{3a+b-\sqrt{3a+b^2-8a^2}}{4}$ is the only proper solution.

In the second equation, since a is greater than b , $\frac{3a-b}{2}$ must be always positive, and therefore the equation is necessarily of the third form. But the roots are possible only when $\left(\frac{3a-b}{4}\right)^2$, is not less than $\frac{a^2}{2}$, that is, when a^2+b^2 is not less than $6ab$, or when $a-b$ is not less than $2\sqrt{ab}$. When the roots are possible, z may be either greater or less than $\frac{3a-b}{4}$, and hence each root gives a proper solution; therefore, $z = \frac{3a-b \pm \sqrt{3a-b^2-8a^2}}{4}$.

Ex. Let $a=40$, and $b=6$. The first term in this case may be assumed either as the greatest or the least. And, first, if z be the greatest, the roots of the equation will be possible, since $(a^2+b^2=) 1636$ is greater than $(6ab=) 1440$. The two values of z are 32 and 25, and the proportionals are either 32, 8, 2, or 25, 15, 9. 2dly, If z be

be assumed the least of the proportionals; the two roots of the equation are possible, but one of them only can be applied, which is 17.635 nearly, and the three proportionals are 17.635, 22.365, and 28.365, nearly, the roots of the equation being incommensurate.

In like manner may the limitations of the other question above mentioned be ascertained.

Though the preceding questions have been so contrived that the answers may be integers, yet in practice it will most commonly happen that they must be surds. When in any question the root of a number which is not a perfect square, is to be extracted, it may be continued in decimals, by the common arithmetical rule, to any degree of accuracy which the nature of the subject may require.

S C H O-

S C H O L I U M.

An equation, in the terms of which two powers only of the unknown quantity are found, and such that the index of the one is double that of the other, may, by the preceding rules, be reduced to a pure equation, and may therefore be resolved by Sect. I. of this chapter.

Such an equation may generally be represented thus,

$$x^{2m} \pm ax^m = \pm b^n$$

Let $x^m = z$, then $z^2 \pm az = \pm b^n$

$$\text{And } x^m (=z) = \mp \frac{a}{2} \pm \sqrt{\frac{a^2}{4} \pm b^n}$$

$$\text{Therefore } x = \sqrt[m]{\mp \frac{a}{2} \pm \sqrt{\frac{a^2}{4} \pm b^n}}$$

E X A M P L E XV.

To find two numbers, of which the product is 100, and the difference of their square roots 3.

Let

Let the less be x ,
the greater is

By question

$$\begin{array}{l|l}
 1 & \frac{100}{x} \\
 2 & \frac{10}{\sqrt{x}} - \sqrt{x} = 3 \\
 3 & 10 - x = 3\sqrt{x} = 3x^{\frac{1}{2}} \\
 4 & x + 3x^{\frac{1}{2}} = 10 \\
 5 & x + 3x^{\frac{1}{2}} + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4} \\
 6 & x^{\frac{1}{2}} + \frac{3}{2} = \frac{7}{2} \text{ and } x^{\frac{1}{2}} = 2 \text{ or } -5 \\
 7 & x = 4 \text{ or } x = 25
 \end{array}$$

If $x=4$ the other number is 25, and this is the proper solution, for x was supposed to be the least. In this case indeed, the negative root of the equation being applied, according to the rules for negative quantities, gives a positive answer to the question, and if $x=25$, the other number is 4.

The same would have been got, by substituting in the general theorem, $m=\frac{1}{2}$, $a=3$, and $b^2=10$; or, if the less number had been called x^2 , the equation would not have had fractional exponents.

C H A P.

C H A P. VI.

Of Indeterminate Problems.

IT was formerly observed, (Chap. III.) that if there are more unknown quantities in a question, than equations by which their relations are expressed, it is indetermined ; or it may admit of an infinite number of answers. Other circumstances, however, may limit the number in a certain manner, and these are various, according to the nature of the problem. The contrivances by which such problems are resolved, are so very different in different cases, that they cannot be comprehended in general rules.

Q

E X-

E X A M P L E I.

To divide a given square number into two parts, each of which shall be a square number.

There are two quantities sought in this question, and there is only one equation expressing their relation ; but it is required also, that they may be rational, which circumstance cannot be expressed by an equation ; another condition therefore must be assumed in such a manner as to obtain a solution in rational numbers.

Let the given square be a^2 , let one of the squares sought be x^2 , the other is $a^2 - x^2$. Let $rx - a$ also be a side of this last square, therefore

$$r^2x^2 - 2rxa + a^2 = a^2 - x^2$$

By transp.

$$r^2x^2 + x^2 = 2rxa$$

Divide by x

$$r^2x + x = 2ra$$

Therefore

$$x = \frac{2ra}{r^2 + 1}$$

$$\text{And } rx - a \left(= \frac{2r^2a}{r^2 + 1} - a \right) \frac{r^2a - a}{r^2 + 1}$$

Let r therefore be assumed at pleasure, and

$\frac{2ra}{r^2+1}$, $\frac{r^2a-a}{r^2+1}$, which must always be rational, will be the sides of the two squares required.

Thus, if $a^2=100$. Then, if $r=3$, the sides of the two squares are 6 and 8, for $36+64=100$.

Also let $a^2=64$. Then if $r=2$, the sides of the squares are $\frac{32}{5}$ and $\frac{24}{5}$; and $\frac{1024}{25} + \frac{576}{25} = \frac{1600}{25} = 64$.

The reason of the assumption of $rx-a$ as a side of the square a^2-x^2 , is that being squared and put equal to this last, the equation manifestly will be simple, and the root of such an equation is always rational.

EXAMPLE II.

To find two square numbers whose difference is given.

Let

Let x^2 and y^2 be the square numbers, and a their difference,

Put $\frac{z+v}{2}=x$, and $\frac{z-v}{2}=y$

$$\frac{z^2 + 2zv + v^2}{4} = x^2$$

$$\frac{z^2 - 2zv + v^2}{4} = y^2$$

$$zv = (x^2 - y^2) = a.$$

If x and y are required only to be rational, then take v at pleasure, and $z = \frac{a}{v}$, whence x and y are known.

But if x and y are required to be whole numbers, Take for z and v any two factors that produce a , and are both even or both odd numbers. And this is possible only where a is either an odd number greater than 1, or a number divisible by 4. Then $\frac{z+v}{2}$ and $\frac{z-v}{2}$ are the numbers sought.

For the product of two odd numbers is odd, and that of two even numbers is divisible by 4. Also, if z and v are both odd
or

or both even, $\frac{z+v}{2}$ and $\frac{z-v}{2}$ must be integers.

Ex. 1. If $a=27$, take $v=1$, then $z=27$; and the squares are 196 and 169. or z may be 9 and $v=3$, and then the squares are 36 and 9.

2. If $a=12$, take $v=2$, and $z=6$; and the squares are 16 and 4.

E X A M P L E III.

To find a sum of money in pounds and shillings, whose half is just its reverse.

Note. The reverse of a sum of money, as 8 l. 12 s. is 12 l. 8 s.

Let x be the pounds, and y the shillings.

The sum required is $20x + y$

Its reverse is $- 20y + x$

Therefore $- \frac{20x+y}{2} = 20y + x$

$$20x + y = 40y + 2x$$

$$18x = 39y$$

$$x : y :: (39 : 18 ::) 13 : 6$$

In

In this equation there are two unknown quantities; and, in general, any two numbers of which the proportion is that of 13 to 6, will agree to it.

But, from the nature of this question, 13 and 6 are the only two that can give the proper answer, viz. 13 l. 6 s. for its reverse 6 l. 13 s. is just its half.

The ratio of x and y is expressed in the lowest integral terms by 13 and 6; any other expression of it, as the next greater 26 and 12, will not satisfy the problem, as 12 l. 26 s. is not a proper notation of money in pounds and shillings.

CHAP.

C H A P. VII.

Demonstration of Theorems by Algebra.

ALGEBRA may be employed for the demonstration of Theorems, with regard to all those quantities concerning which it may be used as an analysis, and from the general method of notation and reasoning, it possesses the same advantages in the one as in the other. The three first sections of this chapter contain some of the most simple properties of *series* which are of frequent use ; and the last, miscellaneous examples of the properties of algebraical quantities and numbers.

I.

I. *Of Arithmetical Series.*

Def. When a number of quantities increase or decrease by the same common difference, they form an *Arithmetical Series*.

Thus, $a, a+b, a+2b, a+3b, \&c. x, x-b, x-2b, \&c.$

Also, 1, 2, 3, 4, 5, 6, &c. And 8, 6, 4, 2, &c.

Prop. In an arithmetical series, the sum of the first and last terms is equal to the sum of any two intermediate terms, equally distant from the extremes.

Let the first term be a , the last x , and b the common difference ; then $a+b$ will be the second, and $x-b$ the last but one, &c.

Thus, $a, a+b, a+2b, a+3b, a+4b, \&c. x, x-b, x-2b, x-3b, x-4b, \&c.$

It is plain, that the terms in the same perpendicular rank are equally distant from the

the extremes, and that the sum of any two in it is $a+x$, the sum of the first and last.

Cor. 1. Hence the sum of all the terms of an arithmetical series, is equal to the sum of the first and last, taken half as often as there are terms.

Therefore if n be the number of terms, and s the sum of the series ; $s = \overline{a+x} \times \frac{n}{2}$.

If $a=0$, then $s = \frac{nx}{2}$.

Cor. 2. The same notation being understood, since any term in the series consists of a , the first term, together with b taken as often as the number of terms preceding it, it follows, that $x = \overline{a+n-1 \times b}$, and hence $s = \overline{2a+n-1 \times b} \times \frac{n}{2}$; or by multiplication, $s = \frac{2an+n^2b-nb}{2}$. Therefore from the first term, the common difference, and number of terms being given, the sum may be found.

Ex. Required the sum of 50 terms of the series 2, 4, 6, 8, &c.

R

s

$$s = \frac{2 \times 2 \times 50 + 50^2 \times 2 - 50 \times 2}{2} = \frac{5100}{2} = 2550.$$

Cor. 3. Of the first term, common difference, sum and number of terms, any three being given, the fourth may be found by resolving the preceding equation ; a , b , s , and n being successively considered as the unknown quantity. In the three first cases the equation is simple, and in the last it is quadratic.

II. Of Geometrical Series.

Def. When a number of quantities increase by the same multiplier, or decrease by the same divisor, they form a *Geometrical Series*. This common multiplier or divisor is called the *common ratio*.

Thus a , ar , ar^2 , &c. a , $\frac{a}{r}$, $\frac{a}{r^2}$, $\frac{a}{r^3}$, &c.

1, 2, 4, 8, &c.

Prop. I. The product of the extremes in a geometrical series, is equal to the product of
of

of any two terms, equally distant from the extremes.

Let a be the first term, y the last, r the common ratio : then the series is,

$$a, ar, ar^2, ar^3, ar^4, \&c.$$

$$y, \frac{y}{r}, \frac{y}{r^2}, \frac{y}{r^3}, \frac{y}{r^4}, \&c.$$

It is obvious, that any term in the upper rank is equally distant from the beginning, as that below it from the end ; and the product of any two such is equal to ay , the product of the first and last.

Prop. II. The sum of a geometrical series wanting the first term, is equal to the sum of all but the last term, multiplied by the common ratio.

For, assuming the preceding notation of a series, it is plain, that

$$\begin{aligned} ar + ar^2 + ar^3, \&c. \dots + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y = \\ = r \times a + ar + ar^2, \&c. \dots + \frac{y}{r^4} + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} \end{aligned}$$

Cor.

Cor. 1. Therefore s being the sum of the series,

$$\overline{s-y} \times r = s - a. \quad \text{And } s = \frac{yr-a}{r-1}$$

Hence s can be found from a , y , and r , and any three of the four being given, the fourth may be found.

Cor. 2. Since the exponent of r in any term is equal to the number of terms preceding it ; hence in the last term its exponent will be $n-1$; the last term therefore $y = ar^{n-1}$, and $s = \frac{ar^n - a}{r-1} = a \times \frac{r^n - 1}{r-1}$. Hence of these four, s , a , r , n , any three being given, the fourth may be found by the solution of equations. If n is not a small number, the cases of this problem will be most conveniently resolved by logarithms ; and of such solutions there are examples in the appendix to this part.

Cor. 3. If the series decreases, and the number of terms is infinite, then according to this notation, a the least term will be 0, and $s = \frac{yr}{r-1}$, a finite sum.

Ex.

Ex. Required the sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, &c. to infinity.

Here $y=1$, and $r=2$. Therefore $s = \frac{1 \times 2}{2-1} = 2$.

What are called in arithmetic, repeating and circulating decimals, are truly geometrical decreasing serieses, and therefore may be summed by this rule.

Thus .333, &c. $= \frac{3}{10} + \frac{3}{100} + \&c.$ is a geometrical series in which $y = \frac{3}{10}$ and $r = 10$,

therefore $s = \frac{yr}{r-1} = \frac{3 \times 10}{10 \times 10 - 1} = \frac{1}{3}$.

Thus also .2424, &c. $= \frac{8}{33}$, for here $y = \frac{24}{100}$

and $r=100$, therefore $s = \frac{24 \times 100}{100 \times 100 - 1} = \frac{24}{99} = \frac{8}{33}$.

III. *Of Infinite Series.*

It was observed, (Chap. I. and IV.) that in many cases, if the division and evolution of compound quantities be actually performed,

formed, the quotients and roots can only be expressed by a series of terms, which may be continued *ad infinitum*. By comparing a few of the first terms, the law of the progression of such a series will frequently be discovered, by which it may be continued without any farther operation. When this cannot be done, the work is much facilitated by several methods; the chief of which is that by the *binomial theorem*.

Theorem.

Any binomial (as $a+b$) may be raised to any power (m) by the following rules.

1. From inspecting a table of the powers of a binomial obtained by multiplication, it appears that the terms without their coefficients are a^m , $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, &c.

2. The coefficients of these terms will be found by the following rule.

Divide

Divide the exponent of a in any term by the exponent of b increased by 1, and the quotient multiplied by the coefficient of that term will give the coefficient of the next following term.

This rule is found, upon trial in the table of powers, to hold universally. The coefficient of the first terms is always 1. and by applying the general rule now proposed, the coefficients of the terms in order will be as follows, 1, m , $m \times \frac{m-1}{2}$, $m \times \frac{m-1}{2} \times \frac{m-2}{3}$, &c. They may be more conveniently expressed thus, 1, $A m$, $B \times \frac{m-1}{2}$, $C \times \frac{m-2}{3}$, $D \times \frac{m-3}{4}$, &c. the capitals denoting the preceeding coefficient. Hence $\overline{a+b}^m = a^m + A m a^{m-1} b + B \times \frac{m-1}{2} \times a^{m-2} b^2 + C \times \frac{m-2}{3} a^{m-3} b^3$ &c. This is the celebrated *binomial theorem*: It is deduced here by induction only, but it may be rigidly demonstrated, though upon principles which do not belong to this place.

Cor.

Cor. 1. As m may denote any number integral or fractional, positive or negative; hence the division, involution and evolution of a binomial, may be performed by this theorem.

Ex. 1. Let $m = \frac{1}{2}$, then $\sqrt{a+b} = a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b + \frac{1}{2} \times -\frac{1}{4} \times a^{-\frac{3}{2}}b^2 +$, &c. This being applied to the extraction of the square root of a^2+x^2 (by inserting a^2 for a and x^2 for b) the same series results as formerly. (Chap. IV.)

Ex. 2. If $\frac{1}{1-r}$ is to be turned into an infinite series, since $\frac{1}{1-r} = 1 \times \sqrt{1-r}^{-1}$, let $a=1$, $b=-r$, and $m=-1$; and the same series will arise as was obtained by division. (Chap. I.)

In like manner $\frac{r^2}{\sqrt{2rz-z^2}} (= r^2 \times \sqrt{2rz-z^2}^{-\frac{1}{2}})$ may be expressed by an infinite series, by supposing $a=2rz$, $b=-z^2$, and $m=-\frac{1}{2}$, and then multiplying that series by r^2 .

Cor.

Cor. 2. This theorem is useful also in discovering the *law* of an infinite series, produced by division or evolution. Thus, the series expressing the square root of $a^2 + x^2$, consists of a , together with a series of fractions, in the numerators of which are the even powers of x , and in the denominators the odd powers of a . The numeral coefficients of the terms of the whole series, as deduced by the theorem, will be :

$$1, + \frac{1}{2 \times 1}, - \frac{1 \times 1}{2.2 \times 1.2}, + \frac{1 \times 1.3}{2.2.2 \times 1.2.3}, -$$

$$\frac{1 \times 1.3.5}{2.2.2.2 \times 1.2.3.4} \&c. \text{ the point being used}$$

(as it often is) to express the product of the numbers between which it is placed. The law of continuation is obvious, and the series may be carried on to any number of terms, without using the theorem. Hence also the coefficient of the n th term is $\frac{1 \times 1.3.5 \&c. \dots (n-2 \text{ terms})}{2^{n-1} \times 1.2.3.4 \&c. (n-1)}$; and it is $+$ if n is an even number, and $-$ if n is odd.

Note, If the binominal is $a+b$, the signs of the terms of any power are all positive: If it is $a-b$, the alternate terms are negative,

tive, beginning at the second. This theorem may be applied to quantities which consist of more than two parts, by supposing them distinguished into two, and then substituting for the powers of these compound parts, their values, to be obtained also, if required, from the theorem. Thus,

$$\overbrace{a+b+c}^2 = \overbrace{a+b+c}^3.$$

S C H O L I U M.

An infinite series may itself be multiplied or divided by another; it may be involved or evolved, and various other operations may be performed upon it which are necessary in the higher parts of algebra. The methods for finding the sum depend upon other principles.

IV. *Properties of Numbers.*

THEOR. I. The sum of two quantities multiplied by their difference is equal to the difference of their squares.

Let

Let the quantities be represented by a and b , then $\overline{a+b} \times \overline{a-b} = a^2 - b^2$, as appears by performing the operation.

COR. If a and b be any two quantities of which the sum may be denoted by s , the difference by d , and their product by p , then the following propositions will be true.

- | | |
|-------------------------------------|------------------------------------|
| 1. $a^2 + b^2 = s^2 - 2p$ | 2. $a^2 - b^2 = sd$ |
| 3. $a^3 + b^3 = s^3 - 3ps$ | 4. $a^3 - b^3 = s^2d - dp$ |
| 5. $a^4 + b^4 = s^4 - 4ps^2 + 2p^2$ | 6. $a^4 - b^4 = s^3d - 2sdp$, &c. |

It is unnecessary to express these propositions in words, and the demonstrations are very easy, by raising $a+b$ to certain powers, and making proper substitutions.

THEOR. II. The sum of any number of terms (n) of the odd numbers 1, 3, 5, &c. beginning with 1, is equal to the square of that number (n).

In the rule for summing an arithmetical series, let $a=1$, $b=2$, and $n=n$, and the sum of

of this series will be $s = \frac{2an + n^2b - nb}{2} = \frac{2n^2}{2} = n^2$. Q. E. D.

THEOR. III. The difference of any two square numbers is equal to the sum of the two roots, together with twice the sum of the numbers in the natural scale between the two roots.

Let the one number be p , and the other $p+n$, the intermediate numbers are $p+1$, $p+2$, ... &c. $p+n-1$. The difference of the squares of the given numbers is $2pn+n^2$; the sum of the two roots is $2p+n$, and twice the sum of the series $\overline{p+1} + \overline{p+2}$... &c. $\overline{p+n-1}$ is (by Cor. 1. 1st Sect. of this chap.) $2s = \overline{2p+n} \times \overline{n-1}$, viz. the sum of the first and last multiplied by the number of terms, and it is plain that $2p+n + \overline{2p+n} \times \overline{n-1} = 2pn+n^2$. Therefore, &c.

LEM. I. Let r be any number, and n any integer, $r^n - 1$ is divisible by $r - 1$.

The quotient will be $r^{n-1} + r^{n-2}$, &c. till the index of r be 0, and then the last term of

of it will be 1 ; for if this series be multiplied by the divisor $r - 1$, it will produce the dividend $r^n - 1$. It will appear also by performing the division, and inserting for n any number.

LEM. 2. Let r be any number, and n any integer odd number, $r^n + 1$ is divisible by $r + 1$. Also, if n is any even number, $r^n - 1$ is divisible by $r + 1$.

The quotient in both cases is $r^{n-1} - r^{n-3} + r^{n-5} - \&c.$ till the exponent of r be 0, and the last term $r^0 = 1$. If this series consist of an odd number of terms, and be multiplied by $r + 1$ the divisor, the product is $r^n + 1$ the dividend. If the series consist of an even number of terms, the product is $r^n - 1$; but it is plain that the number of terms will be odd only when n is odd, and even only when n is even. The conclusion will be manifest by performing the division.

LEM. 3. If r is the root of an arithmetical scale, any number in that scale may be represented in the following manner, $a, b, c,$
 $\&c.$

&c. being the coefficients or digits, $a + br + cr^2 + dr^3 + er^4$, &c.

THEOR. IV. If from any number in the general scale now described, the sum of its digits be subtracted, the remainder is divisible by $r-1$.

The number is $a + br + cr^2 + dr^3$, &c. and the sum of the digits is $a + b + c + d$, &c. Subtracting the latter from the former, the remainder is $br - b + cr^2 - c + dr^3 - d$, &c. $= b \times \overline{r-1} + c \times \overline{r^2-1} + d \times \overline{r^3-1}$ &c. But (by Lem. I.) $r^n - 1$ is divisible by $r-1$, whatever integer number n may be, and therefore any multiple of $r^n - 1$ is also divisible by $r-1$: Hence each of the terms, $b \times \overline{r-1}$, $c \times \overline{r^2-1}$, &c. is divisible by $r-1$, and therefore the whole is divisible by $r-1$.

COR. I. Any number, the sum of whose digits is divisible by $r-1$, is itself divisible by $r-1$. Let the number be called N , and the sum of the digits D ; then by this Prop. $N-D$ is divisible by $r-1$, and D is supposed

posed to be divisible by $r-1$; therefore it is plain that N must also be divisible by $r-1$.

COR. 2. Any number, the sum of whose digits is divisible by an aliquot part of $r-1$, is also divisible by that aliquot part. For, let N and D denote as before ; and since $N-D$ (Theor. 4.) is divisible by $r-1$, it is also divisible by an aliquot part of $r-1$; but D is divisible by an aliquot part of $r-1$, therefore N is also divisible by that aliquot part.

COR. 3. This theorem, with the corollaries, relates to any scale whatever. It includes therefore the well known property of 9 and of 3 its aliquot part, in the decimal scale ; for, since $r=10$, $r-1=9$.

THEOR. V. In any number, if from the sum of the coefficients of the odd powers of r the sum of the coefficients of the even powers be subtracted, and the remainder added to the number itself, the sum will be divisible by $r+1$.

In

In the number $a+br+cr^2+dr^3+er^4+fr^5$, &c. the sum of the coefficients of the odd powers of r is $b+d+f$, &c. the sum of the coefficients of the even powers of r is $a+c+e$, &c. If the latter sum be subtracted from the former, and the remainder added to the given number, it makes $br+b+cr^2-c+dr^3+d+er^4-e+fr^5+f$, &c. $= b \times \overline{r+1} + c \times \overline{r^2-1} + d \times \overline{r^3+1} + e \times \overline{r^4-1} + f \times \overline{r^5+1}$, &c. But (by Lem. 2.) $r+1, r^2-1, r^3+1$, &c. are each divisible by $r+1$, and therefore any multiples of them are also divisible by $r+1$, hence the whole number is divisible by $r+1$.

Cor. 1. If the difference of the sum of the even digits, and the sum of the odd digits of any number be divisible by $r+1$, the number itself is divisible by $r+1$.

Let the sum of the even digits (that is, the coefficients of the odd powers of r) be D , the sum of the odd digits be d , and let the number be N . Then by the theorem $N+D-d$ is divisible by $r+1$, and it is supposed that $D-d$ is divisible by $r+1$; therefore N is divisible by $r+1$.

Cor. 2. In like manner, if $D-d$ is divisible by an aliquot part of $r+1$, N will be divisible by that aliquot part.

Cor.

Cor. 3. If a number want all the odd powers of r , or if it want all the even powers of r , and if the sum of its digits be divisible by $r+1$, that number is divisible by $r+1$.

Cor. 4. In the common scale $r+1 \pm 11$, which therefore will have the properties mentioned in this theorem, and the corollaries. Thus, in the number 64834, the sum of the even digits is 7, the sum of the odd digits is 18, and the difference is 11, a number divisible by 11, the given number therefore (Cor. 1.) is divisible by 11. Thus also, the sum of the digits of 7040308 is divisible by 11, and therefore the number is divisible by 11. (Cor. 3.)

S C H O L I U M.

These Theorems relate to any scale whatever, and therefore the properties of $r-1$ in Theor. 4. would in a scale of *eight* be-

T long

long to *seven*, and those in Theor. 5. to *nine*. If *twelve* was the root of the scale, the former properties would belong to *eleven*, and the latter to *thirteen*.

APPEN-

APPENDIX to Part I.

ALGEBRA may be employed in expressing the relations of magnitude in general, and in reasoning with regard to them. It may be used in deducing not only the relations of number, but also those of extension, and hence, those of every species of quantity expressible by numbers or extended magnitudes. In this appendix are mentioned some examples of its application to other parts of mathematics, to physics, and to the practical calculations of business. The principles and suppositions peculiar to these subjects, which are necessary in directing both the algebraical operations, and the conclusions to be drawn from them, are here assumed as just and proper.

I.

Algebra has been successfully applied to almost every branch of mathematics ; and the principles of these branches are often advantageously introduced into algebraical calculations.

The application of it to *geometry* has been the source of great improvements in both these sciences ; on account of its extent and importance it is here omitted, and the principles of it are more particularly explained in the third part of these elements.

In this place shall be given an example of the use of logarithms, in resolving certain algebraical questions.

Note. When Logarithms are used, let (*l.*) denote the logarithm of any quantity before which it is placed.

Ex. To find the number of terms of a geometrical series, of which the sum is 511, the first term 1, and the common ratio 2.

From Sect. 2. Chap. 7. it appears that $s = \frac{ar^n - a}{r - 1}$, and in this problem, *s*, *r*, and *a*

are

are given, and n is to be found. By reducing the equation $r^n = \frac{s \times r - 1 + a}{a}$ and from

the known property of logarithms $n \times l.r = l.s \times r - 1 + a - l.a$, and $n = \frac{l.s \times r - 1 + a - l.a}{l.r}$.

But here $s = 511$, $a = 1$, $r = 2$, and $n = \frac{l.512}{l.2} = \frac{2.7092700}{0.3010300} = 9$.

In like manner may any such equation be resolved, when the only unknown quantity is an exponent, and when it is the exponent only of one quantity.

Ex. 2. An equation of the following quadratic form $a^x \pm 2ba^x = \pm c$ may be resolved by logarithms. 1st, by scholium of chap. 5. $a^x = \mp b \pm \sqrt{b^2 \pm c}$. And then x is discovered in the same manner as in the preceding example. Thus, let $a = 2$, $b = 10$, and $c = 96$ and the equation $2^{2x} - 20 \times 2^x = -96$. 1st. $2^x = 10 \pm \sqrt{4} = 12$ or 8 . If $2^x = 8$ then $x = \frac{l.8}{l.2} = 3$ and $2^6 - 20 \times 2^3 = -96$ is a true equation. If $2^x = 12$ then $x = \frac{l.12}{l.2} = \frac{1.0791812}{0.3010300} = 3.5849$, and this number being inserted for

x

x in the given equation, by means of logarithms, will answer the conditions.

— Ex. 3. The sum of 2000 l. has been out at interest for a certain time, and 500 l. has been at interest double of that time, the whole arrear now due reckoning 4 per cent. compound interest, is 6000 l. What were the times?

By the rules in the third part of this appendix for compound interest, it is plain that if $R=1.04$, and the time at which the 2000 l. is at interest be x , the arrear of it will be $2000 \times R^x$. The arrear of the 500 l. is $500 \times R^{2x}$, hence $500 \times R^{2x} + 2000 \times R^x = 6000$. This resolved gives $R^x = 2$ and $x = \frac{\log 2}{\log R} = 17.67$, + nearly, that is 17 years and 8 months nearly, and the double is 35 years and 4 months; which answer the conditions.

II. *Application of Algebra to Physics.*

Physical quantities which can be divided into parts, that have proportions to each other, the same as the proportions of lines

to

to lines, or of numbers to numbers, may be expressed by lines and numbers, and therefore by algebraical quantities. Hence these mathematical notations may be considered as the measures of such physical quantities; they may be reasoned upon, according to the principles of algebra, and from such reasonings, new relations of the quantities which they represent, may be discovered.

In those branches of natural philosophy, therefore, in which the circumstances of the phaenomena can be properly expressed by numbers, or geometrical magnitudes, algebra may be employed, both in promoting the investigation of physical laws by experience, and also, in deducing the necessary consequences of laws investigated and presumed to be just.

It is to be observed likewise, that if various hypotheses be assumed, concerning physical quantities, without regard to what takes place in nature, their consequences may be demonstratively deduced, and thus a science may be established, which may be properly called *mathematical*. The use of algebra

algebra in this science, which is sometimes called *Theoretical Mechanics*, is obvious from the principles already laid down.

In conducting these inquiries, it is to be observed, that, for the sake of brevity, the language of algebraical operations is often used, with regard to physical quantities themselves ; though it is always to be understood, that, in strict propriety, it can be applied only to the mathematical notations of these quantities.

Before illustrating this application of algebra by examples, it may be proper to explain a method of stating the proportion of variable quantities, and reasoning with regard to it, which is of general use in natural philosophy.

1. *Of the proportion of variable Quantities.*

Mathematical quantities are often so connected, that when the magnitude of one is varied, the magnitudes of the others are varied, according to a determined rule. Thus, if two straight lines, given in position,

tion, intersect each other; and, if a straight line, cutting both, moves parallel to itself, the two segments of the given lines between their intersection, and the moving line, however varied, will always have the same proportion. Thus also, if an ordinate to the diameter of a parabola move parallel to itself, the absciss will be increased or diminished, in proportion as the square of the ordinate is increased or diminished.

In like manner may algebraical quantities be connected. If x, y, z , &c. represent any variable quantities, while a, b, c , represent such as are constant or invariable, then an equation containing two or more variable quantities, with any number of constant quantities, will exhibit a relation of variable quantities, similar to those already mentioned. Thus, if $ax = by$, then $x : y :: b : a$, that is, x has a constant proportion to y , in whatever way these two quantities may be varied. Likewise, if $xy^2 = a^2b$, then $y^2 : a^2 :: b : x$, or $y^2 : \frac{1}{x} :: a^2 : \frac{1}{b}$, that is, y^2 has a constant proportion to the reciprocal of x , or, y^2 is increased in the same proportion as x is diminished,

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minished, and conversely. It is necessary to premise the following definitions.

D E F I N I T I O N S.

Let there be any number of variable quantities $X, Y, Z, V, \&c.$ connected in such a manner, that, when X becomes x , $Y, Z, V, \&c.$ become respectively $y, z, v, \&c.$ And let $a, b, c, \&c.$ represent any constant quantities, whether given or unknown. Then

1. If two variable quantities X and Y are so connected, that, whatever be the values of x and y , $X : x :: Y : y$, this proportion is expressed thus, $X = Y$, and X is said to be *directly as* Y , or shortly, X is said to be *as* Y .

2. If two variable quantities X and Y are so connected, that $X : x :: y : Y$, or $X : x :: \frac{1}{Y} : \frac{1}{y}$; their relation is thus expressed, $X = \frac{1}{Y}$; and X is said to be *inversely*, or *reciprocally as* Y .

3. If X, Y, Z , are three variable quantities, so connected that $X : x :: YZ : yz$, their
relation

relation is so expressed, $X = YZ$, and X is said to be *directly as* Y and Z , *jointly*; or X is said to be *as* Y and Z .

4. If any number of variable quantities as X , Y , Z , V , &c are so connected, that $XY : xy :: \frac{rZ}{V} : \frac{yz}{v}$; then $XY = \frac{rZ}{V}$, and XY is said to be *directly as* YZ , and *inversely as* V , or more explicitly, X and Y *jointly*, are *directly as* Y and Z *jointly*, and *inversely as* V .

In like manner are other combinations of variable quantities denoted and expressed.

It is to be observed also, the same definitions take place, when the variable quantities are multiplied or divided by any constant quantities. Thus, if $aX : ax :: \frac{b}{Y} : \frac{b}{y}$ then $aX = \frac{b}{Y}$, &c.

5. Let the preceding notation of proportion be called a *proportional equation**, the equa-

* These terms are used only with a view to give more precision to the ideas of beginners. In order to avoid the ambiguity in the meaning of the sign $=$, some writers employ the character $\propto$, to denote constant proportion; but this is seldom necessary, as the quantities compared are generally of different kinds, and the relation expressed is sufficiently obvious. See Emerson's Mathematics, Vol. 1.

equations formerly treated of being in this place, for the sake of distinction, called *absolute*.

COR. Every absolute equation, containing more than one variable quantity, may be considered as a proportional equation; and, in a proportional equation, if at any particular corresponding values of the variable quantities, the equation becomes absolute, it will be universally absolute.

PROP. I. If one side of a proportional equation be either multiplied or divided by any constant quantity, it will continue to be true. Thus, if $X = \frac{1}{Y}$, then $X = \frac{a^2}{bY}$. For, since $X = \frac{1}{Y}$ (Def. 3.) $X : x :: \frac{1}{Y} : \frac{1}{y}$, it follows (Chap. 2.) that $X : x :: \frac{a^2}{bY} : \frac{a^2}{by}$, therefore (def. 4.) $X = \frac{a^2}{bY}$.

PROP. 2. If the two sides of a proportional equation be both multiplied, or both divided by the same quantity, it will continue to be true.

1st, If the quantity be constant, it is manifest from Prop. 1.

2^d,

2d, If the quantity be variable, let $X=Y$, and Z a variable quantity, then $XZ=YZ$. For, since $X=Y$, (Def. 2.) $X:x::Y:y$; multiply the antecedents by Z , and the consequents by z , then $XZ:xz::YZ:yz$, therefore (Def. 5.) $XZ=YZ$. In like manner, if $X=Y$, $\frac{X}{Z}=\frac{Y}{Z}$.

COR. Any variable quantity, which is a factor of one side of a proportional equation, may be made to stand alone. Thus, if $XY=\frac{Z}{V}$, then $X=\frac{Z}{VY}$; also, $Z=XYV$; and $Y=\frac{Z}{XV}$, and also $V=\frac{Z}{XY}$, &c. Hence, also, if one side of a proportional equation be divided by the other, the quotient is a constant quantity, viz. 1.

PROP. 3. If two proportional equations have a common side, the remaining two sides will form a proportional equation. Also, that common side will be as the sum or difference of the other two.

Thus, if $X=Y$, and $Y=Z$, then $X=Z$. For $X:x::Y:y$, and $Y:y::Z:z$, therefore multiplying these ratios, $XY:xy::YZ:yz$, and by dividing

dividing antecedents and consequents, $X : x :: Z : z$, therefore (Def. 2.) $X = Z$.

Likewise, if $X = Y$, and $Y = Z$, $Y = X \pm Z$. For, since $X : x :: Y : y :: Z : z$. (Chap. 2.) $Y : y :: X \pm Z : x \pm z$, therefore Def. 5. $Y = X \pm Z$.

COR. Hence, one side of a proportional equation, will be *as* the sum, or *as* the difference of the two sides ; and the sum of the two sides will be *as* their difference. Thus, if $X = Y + Z$, then $X = X + Y + Z$ and $X = X - Y - Z$, and also $X + Y + Z = X - Y - Z$.

PROP. 4. If the two sides of a proportional equation be respectively multiplied or divided by the two sides of any other proportional equation, the products or quotients will form a proportional equation.

Thus, if $X = Y$, and $Z = V$, then $XZ = YV$. For, since $X : x :: Y : y$, and $Z : z :: V : v$, by multiplying these proportions (Chap. 1. 2.) $XZ : xz :: YV : yv$, therefore (Def. 5.) $XZ = YV$. In like manner in the case of division.

COR. 1. The two sides of a proportional equation may be raised to any power, or, any root may be extracted out of both, and the equation will continue to be true.

Thus,

Thus, if $X=Y$, then $X^m=Y^m$; for, since $X=Y$, $X:x::Y:y$, and therefore $X^m:x^m::Y^m:y^m$; therefore $X^m=Y^m$. And, if $x=y$ also $X^m=Y^m$.

COR. 2. If two proportional equations have a common side, that side will be *as* the square root of the product of the other two. Thus, if $X=Y$, and $Y=Z$, by this prop. $Y^2=XZ$, and (Cor. 1.) $Y=\sqrt{XZ}$. Hence also, in this case, $\sqrt{XZ}=X\pm Z$; for (prop. 3.) $Y=X\pm Z$.

COR. 3. If one side of a proportional equation be a factor of a side of another proportional equation, the remaining side of the former may be inserted in the latter, in place of that factor. Thus, if $X=ZY$, and $Z=\frac{1}{V}$, then $X=\frac{r}{V}$, as appears by multiplying the two equations, and dividing by Z .

PROP. 5. Any proportional equation may be made absolute, by multiplying one side by a constant quantity.

Thus, if $X=Y$, then let two particular corresponding values of these variable quantities

ties be assumed as constant, and let them be a and b , then $X : a :: Y : b$, and $Xb = aY$, or $X = Y \times \frac{a}{b}$, an absolute equation.

S C H O L I U M.

1. If there be two variable physical quantities, either of the same, or of different kinds, which are so connected, that, when the one is increased or diminished, the other is increased or diminished in the same proportion; or, if the magnitudes of the one, in any two situations, have the same ratio to each other, as the magnitudes of the other in the corresponding situations, the relation of the mathematical measures of these quantities may be expressed by a proportional equation, according to Def. 1.

2. If two variable physical quantities be so connected, that the one increases in the same proportion as the other is diminished, and conversely; or, if the magnitudes of the one, in any two situations, be reciprocally proportional to the magnitudes of the other,
in

in the corresponding situations, the relation of their measures may be expressed by a proportional equation, according to Def. 2.

3. If three variable physical quantities are so connected, that one of them is increased or diminished, in proportion as both the others are increased or diminished; or, if the magnitudes of one of them in any two situations, have a ratio, which is compounded of the ratios of the magnitudes of the other two, in the corresponding situations; the relation of the measures of these three may be expressed by a proportional equation, according to Def. 3.

4. In like manner may the relations of other combinations of physical quantities be expressed, according to Def. 4. And when these proportional equations are obtained, by reasoning with regard to them, according to the preceding propositions, new relations of the physical quantities may be deduced.

2. *Examples of Physical Problems.*

The use of algebra, in natural philosophy, may be properly illustrated by some

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examples

examples of physical problems. The solution of such problems must be derived from known physical laws, which, though ultimately founded on experience, are here assumed as principles, and reasoned upon mathematically. The experiments by which the principles are ascertained admit of various degrees of accuracy ; and on the degree of physical accuracy in the principles will depend the physical accuracy of the conclusions mathematically deduced from them. If the principles are inaccurate, the conclusions must, in like manner, be inaccurate ; and, if the limits of inaccuracy in the principles can be ascertained, the corresponding limits, in the conclusions derived from them, may likewise be calculated.

E X A M P L E I.

Let a glass tube, 30 inches (a) long, be filled with mercury, excepting 8 inches (b) ; and let it be inverted. as in the Torricellian experiment, so that the 8 inches of common air may rise to the top : It is required to find at what height the mercury will remain suspended, the mercury in the barometer

rometer being at that time 28 inches (d) high.

The solution of this problem depends upon the following principles :

1. The pressure of the atmosphere is measured by the column of mercury in the barometer ; and the elastic force of the air, in its natural state, which resists this pressure, is therefore measured by the same column.

2. In different states, the elastic force of the air is reciprocally as the spaces which it occupies.

3. In this experiment, the mercury which remains suspended in the tube, together with the elastic force of the air in the top of it, being a counterbalance to the pressure of the atmosphere, may therefore be expressed by the column of mercury in the barometer.

Let the mercury in the tube be x inches, the air in the top of it occupies now the space $a-x$; it occupied formerly b inches, and its elastic force was d inches of mercury : Now, therefore, the force must be

$$(a-x$$

$(a-x : b :: d :) \frac{bd}{a-x}$ inches (2.). Therefore

(3.) $x + \frac{bd}{a-x} = d$. This reduced, and putting

$a+d=2m$ the equation is $x^2 - 2mx = bd - ad$

This resolved gives $x = m \pm \sqrt{m^2 + bd - ad}$.

In numbers - - $x = 44$ or 14 .

One of the roots 44 is plainly excluded in this case, and the other 14 is the true answer. If the column of mercury x , suspended in the tube, were a counterbalance to the pressure of the atmosphere, expressed by the height of the barometer d , together with the measure of the elastic force of b inches of common air in the space $x-a$, that is, if $x = d + \frac{bd}{x-a}$, or $x - \frac{bd}{x-a} = d$, the equation will be the same as before, and the root 44 would be the true answer. But the experiment in this question does not admit of such a supposition.

E X A M P L E II.

The distance of the earth and moon (d,) and their quantities of matter (t, l,) being given, to find the point of equal attraction, between them.

Let

Let the distance of the point from the earth be x : Its distance from the moon will be therefore $d-x$. But gravitation is *as the matter directly*, and *as the square of the distance inversely*; therefore the earth's attraction is *as* $\frac{t}{x^2}$; and the moon's attraction is *as* $\frac{l}{(d-x)^2}$. But these are here equal; therefore,

$$\frac{t}{x^2} = \frac{l}{(d-x)^2} \text{ and } \frac{\sqrt{t}}{x} = \frac{\sqrt{l}}{d-x}$$

This equation reduced gives $x = \frac{d\sqrt{t}}{\sqrt{t} + \sqrt{l}}$

Or mult. numerator and denominator by $\sqrt{t} - \sqrt{l}$ } $x = \frac{dt - d\sqrt{tl}}{t - l}$

In round numbers, let $d=60$ semidiameters of the earth, $t=40$, $l=1$, then $x=52$ semidiameters nearly. There is another point beyond the moon at which the attractions are equal, and it would be found by putting the square root of $(d-x)^2$ to be $x-d$, which, in this case, would be a positive quantity; and then $x = \frac{dt + d\sqrt{tl}}{t-l} = 72$ nearly. If the quantities had been multiplied

plied before extracting the square roots, the adjected quadratic would have given the same two roots.

E X A M P L E III.

Let a stone be dropt into an empty pit; and let the time from the dropping of it to the hearing the sound from the bottom be given: To find the depth of the pit.

Let the given time be a ; let the fall of a heavy body in the 1st second of time (16.122 feet) be b : also, let the motion of sound in a second (1142 feet) be c .

Let the time of the stone's fall be	1	x
The time in which the sound of it moves to the top is	2	$a - x$
The descent of a falling body is as the square of the time, therefore the depth of the pit is ($1^2 : x^2 :: b :$)	3	bx^2
The depth from the motion of sound is also	4	$ca - cx$
Therefore 3 and 4	5	$bx^2 = ca - cx$

This equation being resolved, gives the value of x , and from it may be got bx^2 or $ca - cx$, the depth of the pit.

If

If the time is 10" then $x=8.885$ nearly, and the depth is 1273 feet.

There are several circumstances in this problem which render the conclusion inaccurate.

1. The values of c and b , on which the solution is founded, are derived from experiments, which are subject to considerable inaccuracies.

2. The resistance of the air has a great effect in retarding the descent of heavy bodies, when the velocity becomes so great as is supposed in this question ; and this circumstance is not regarded in the solution.

3. A small error, in making the experiment to which this question relates, produces a great error in the conclusion. This circumstance is particularly to be attended to in all physical problems ; and, in the present case, without noticing the preceding imperfections, an error of half a second, in estimating the time, makes an error of above 100 feet in the expression of the depth of the pit.

III. *Of Interest and Annuities.*

The application of algebra to the calculation of interests and annuities, will furnish proper examples of its use in business. Algebra cannot determine the propriety or justice of the common suppositions on which these calculations are founded, but only the necessary conclusions resulting from them.

Notation.

In the following theorems let p denote any principal sum of which 1l. is the unit, t the time during which it bears interest, of which 1 year shall be the unit, r the rate of interest of 1 l. for 1 year, and let s be the amount of the principal sum p with its interest for the time t at the rate r .

I. *Of Simple Interest.*

$s = p + ptr$, and of these four, s, p, t, r , any three being given, the fourth may be found by resolving a simple equation.

The

The foundation of the canon is very obvious; for the interest of 1 l. in one year is r , for t years it is tr , and for p pounds it is ptr ; the whole amount of principal and interest must therefore be $p + ptr = s$.

II. *Of Compound Interest.*

When the simple interest at the end of every year is supposed to be joined to the principal sum, and both to bear interest for the following year, money is said to bear compound interest. The same notation being used, let $1 + r = R$. Then $s = pR^t$.

For the simple interest of 1 L. in a year is r , and the new principal sum therefore which bears interest during the second year is $(1 + r) = R$; the interest of R for a year is rR , and the amount of principal and interest at the end of the 2d year, is $R + rR = R \times \overline{1 + r} = R^2$. In like manner, at the end of the 3d year it is R^3 , and at the end of t years it is R^t , and for the sum p it is $pR^t = s$.

Cor. 1. Of these four p , R , t , s any three being given the 4th may be found. When

Y

t

t is not very small, the solution will be obtained most conveniently by logarithms. When R is known r may be found, and conversely.

Ex. If 500 $l.$ has been at interest for 21 years, the whole arrear due, reckoning $4\frac{1}{2}$ per cent. compound interest, is 1260.12 $l.$ or 1260 $l.$ 2s. 5d. In this case $p=500$, $R=1.045$ and $t=21$ and $s=1260.12$, and any one of these may be derived by the theorem from the others being known. Thus, to find s ; $l.R^t=t \times l.R=21 \times 0.0191163=0.4014423$, therefore $R^t=2.520242$ and $s=(pR^t)=500 \times 2.520242=1260.121$.

Cor. 2. The present worth of a sum (s) in reversion that is payable after a certain time t is found thus. Let the present worth be x , then this money improved by compound interest during t produces xR^t , which must be equal to s , and if $xR^t=s$, $x=\frac{s}{R^t}$.

Cor. 3. The time in which a sum is doubled at compound interest will be found thus. $pR^t=2p$ and $R^t=2$ and $t=\frac{l. 2}{l. R}$; thus, if the

the rate is 5 per cent. $r=.05$ and $\frac{L.2}{L.1.05} = \frac{0.3010300}{0.0211893} = 14.2066$, that is 14 years and 75 days nearly.

S C H O L I U M.

Many other suppositions might be made with regard to the improvement of money by compound interest. The interest might be supposed to be joined to the capital, and along with it to bear interest at the end of every month, at the end of every day, or even at the end of every instant, and suitable calculations might be formed; but these suppositions, being seldom used in practice, are omitted.

III. *Of Annuities.*

An annuity is a payment made annually for a certain term of years, and the chief problem, with regard to it is, 'to determine
its

‘ its present worth.’ The supposition on which the solution proceeds is, that the money received by the seller, being improved by him in a certain manner during the continuance of the annuity, amounts to the same sum as the several payments received by the purchaser, improved in the same manner. The suppositions with regard to the improvement may be various. What is called the method of simple interest, in which simple interest only is reckoned upon the purchase money, and simple interest on each annuity from the time of payment, is so manifestly unequitable, as to be universally rejected ; and the supposition which is now generally admitted in practice, is the highest improvement possible on both sides, viz. by compound interest. As the taking compound interest is prohibited by law, the realizing of this supposed improvement requires punctual payment of interest, and therefore, the interest in such calculations is usually made low. Even with this advantage, it can hardly be rendered effectual in its full extent ; it is however universally acquiesced

acquiesced in, as the most proper foundation of general rules; and when peculiar circumstances require any different hypothesis, a suitable calculation may be made.

Let then the annuity be called a , and let p be the present worth of it or purchase money, t the time of its continuance, and let the other letters denote as formerly.

The seller, by improving the price received p , at compound interest, at the time the annuity ceases, has pR^t .

The purchaser is supposed to receive the first annuity a at the end of the first year, which is improved by him for $t-1$ years; it becomes therefore (Th. 2.) aR^{t-1} .

He receives the 2d annuity at the end of the 2d year, and when improved $t-2$, it becomes aR^{t-2} .

The third annuity becomes aR^{t-3} , &c.

The last annuity is simply a , therefore the whole amount of the improved annuities is the geometrical series $a + aR + aR^2$, &c. . . aR^{t-1} . The sum of this series, by Chap. 7. Sect. 2. is $a \times \frac{R^t - 1}{R - 1} = a \times \frac{R^t - 1}{r}$.

But,

But, from the nature of the problem,

$$pR^t = a \times \frac{R^t - 1}{r}, \text{ and hence } p = a \times \frac{R^t - 1}{rR^t} = a \times \frac{1 - \frac{1}{R^t}}{r}.$$

The same conclusion results from calculating the present worth of the several annuities, considered as sums payable in reversion.

Cor. 1. Of these four p , a , R , t , any three being given, the fourth may be found, by the solution of equations; t is found easily by logarithms, R or r can be found only by resolving an affected equation of the t order.

Cor. 2. If an annuity has been unpaid for the term t , the arrear, reckoning compound interest, will be $a \times \frac{R^t - 1}{r}$.

Cor. 3. The present worth of an annuity in reversion, that is to commence after a certain time (n), and then to continue t years, is

is found by subtracting the present worth for n years from the present worth for $n+t$ years, and then ;

$$p = a \times \frac{R^t - 1}{rR^{t+n}} = a \times \frac{1 - \frac{1}{R^t}}{rR^n}.$$

Also of R, t, n, a, p , any four being given, the fifth may be found.

Cor. 4. If the annuity is to continue for ever, then $R^t - 1$ and R^t may be considered as the same ; and $p = a \times \frac{R^t - 1}{rR^t} = \frac{a}{r}$.

Cor. 5. A perpetuity in reversion (by Cor. 3.) since $R^t - 1 = R^t$, is $p = \frac{a}{rR^n}$.

Prob. When 12 years of a lease of 21 were expired, a renewal for the same term was granted for 1000 l. ; 8 years are now expired, and for what sum must a corresponding renewal be made, reckoning 5 per cent. compound interest ?

From the first transaction the yearly profit rent must be deduced ; and from this the proper fine in the second may be computed.

In

In the first bargain, an annuity in reversion for 12 years, to commence 9 years hence, was sold for 1000 l. the annuity will therefore be found by Cor. 3. in which all the quantities are given, but $a = p \times \frac{rR^n}{1 - \frac{r}{R^t}}$.

and by inserting numbers, viz. $p=1000$, $t=12$, $n=9$, $r=.05$, and $R=1.05$; and working by logarithms $a=175.029=175 \text{ l.} - 7 \text{ d.}$

Next, having found a , the second renewal is made by finding the present worth of the annuity a in reversion, to commence 13 years hence, and to last 8 years. In the canon (Cor. 3.) insert for a , 175.029 and let $t=8$, $n=13$, and $r=.05$ as before. $p=599.93=599 \text{ l. } 18 \text{ s. } 6 \text{ d. } \frac{1}{2}$. The fine required.

As these computations often become troublesome, and are of frequent use, all the common cases are calculated in tables, from which the value of any annuity, for any time, at any interest, may easily be found.

It is to be observed also, that the preceding rules are computed on the supposition of the annuities being paid yearly; and therefore, if they be supposed to be paid
half

half yearly, or quarterly, the conclusions will be somewhat different, but they may easily be calculated on the preceding principles.

The calculations of life annuities, depend partly upon the principles now explained, and partly on physical principles, from the probable duration of human life, as deduced from bills of mortality.

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shall yearly or quarterly, and calculations will
be somewhat different, but they may easily
be calculated on the preceding principles.
The calculations of the annuities depend
partly upon the principles now explained,
and partly on physical principles, from the
probable duration of human life, as deter-
mined from bills of mortality, &c. &c. &c.
The first part of the calculation is to find
the present value of a sum of money, to be
received at a future time, when it is
certain that it will be received. This is
done by dividing the sum by the power of
the interest, raised to the power of the time.
The second part is to find the present value
of a sum of money, to be received at a
future time, when it is uncertain whether
it will be received or not. This is done by
multiplying the present value of the sum, if
received, by the probability of its being
received. The probability of its being
received is found from bills of mortality,
&c. &c. &c. The third part is to find the
present value of a sum of money, to be
received at a future time, when it is
uncertain whether it will be received or not,
and when the sum is to be paid in
instalments. This is done by finding the
present value of each instalment, and then
adding them together. The fourth part is
to find the present value of a sum of money,
to be received at a future time, when it is
uncertain whether it will be received or not,
and when the sum is to be paid in
instalments, and when the interest is
not compounded annually, but at some
other period. This is done by finding the
present value of each instalment, and then
adding them together, and then multiplying
the sum by the power of the interest, raised
to the power of the time, divided by the
power of the interest, raised to the power of
the time, divided by the power of the interest,
raised to the power of the time, divided by
the power of the interest, raised to the power
of the time, &c. &c. &c.

E L E M E N T S
O F
A L G E B R A.

P A R T II.

Of the general properties and resolution of
EQUATIONS of all Orders.

C H A P. I.

*Of the Origin and Composition of Equations ;
and of the Signs and Coefficients of their
Terms.*

IN order to resolve the higher orders of equations, and to investigate their general affections, it is proper first to consider their origin from the combination of inferior equations.

As it would be impossible to exhibit particular rules for the solution of every order
of

of equations, their number being indefinite; there is a necessity of deducing rules from their general properties, which may be equally applicable to all.

In the application of algebra to certain subjects, and especially to geometry, there may be an opposition in the quantities, analogous to that of addition and subtraction, which may therefore be expressed by the signs $+$ and $-$. Hence these signs may be understood by abstraction, to denote contrariety in general; and therefore, in this method of treating of equations, negative roots are admitted as well as positive. In many cases the negative will have a proper and determinate meaning; and when the equation relates to magnitude only, where contrariety cannot be supposed to exist, these roots are neglected, as in the case of quadratic equations formerly explained. There is besides this advantage in admitting negative roots, that both the properties of equations from which their resolution is obtained, and also those which are useful in the many extensive applications of algebra, become

become more simple and general, and are more easily deduced.

In this general method, all the terms of any equation are brought to one side, and the equation is expressed by making them equal to 0. Therefore, if a root of the equation be inserted instead of (x) the unknown quantity, the positive terms will be equal to the negative, and the whole must be equal to 0.

DEF. When any equation is put into this form, the term in which (x) the unknown quantity, is of the highest power, is called the *First*; that in which the index of x is less by 1, is the *Second*, and so on, till the last into which the unknown quantity does not enter, and which is called the *Absolute Term*.

Prop. I. If any number of equations be multiplied together, an equation will be produced, of which the dimension* is equal to the

* The term *dimension*, in this treatise, is used in senses somewhat different, but so as not to create any ambiguity. In this chapter it means either the order of an equation, or the number denoting that order, which was formerly defined to be the highest exponent of the unknown quantity in any term of the equation.

the sum of the dimensions of the equations multiplied.

If any number of simple equations be multiplied together, as $x-a=0$, $x-b=0$, $x-c=0$, &c. it is obvious, that the product will be an equation of a dimension, containing as many units as there are simple equations. In like manner, if higher equations are multiplied together, as a cubic and a quadratic, one of the fifth order is produced, and so on.

Conversely. An equation of any dimension is considered as compounded either of simple equations, or of others, such that the sum of their dimensions is equal to the dimension of the given one. By the resolution of equations these inferior equations are discovered, and by investigating the component simple equations, the roots of any higher equation are found.

Cor. 1. Any equation admits of as many solutions, or has as many roots, as there are simple equations which compose it, that is, as there are units in the dimension of it.

Cor.

Cor. 2. And conversely, no equation can have more roots than the units in its dimension.

Cor. 3. Imaginary or impossible roots must enter an equation by pairs; for they arise from quadratics, in which both the roots are such.

Hence also, an equation of an even dimension may have all its roots, or any even number of them impossible, but an equation of an odd dimension must at least have one possible root.

Cor. 4. The roots are either positive or negative, according as the roots of the simple equations, from which they are produced, are positive or negative.

Cor. 5. When one root of an equation is discovered, one of the simple equations is found, from which the given one is compounded. The given equation, therefore, being divided by this simple equation, will give an equation of a dimension lower by 1. Thus, any equation may be depressed as
many

From this table it is plain,

1. That in a complete equation the number of terms is always greater by unit than the dimension of the equation.

2. The coefficient of the first term is 1.

The coefficient of the second term is the sum of all the roots (a, b, c, m , &c.) with their signs changed.

The coefficient of the third term is the sum of all the products that can be made by multiplying any two of the roots together.

The coefficient of the fourth term is the sum of all the products which can be made by multiplying together any three of the roots with their signs changed; and so of others.

The last term is the product of all the roots, with their signs changed.

3. From induction it appears, that in any equation (the terms being regularly arranged as in the preceding example) there are as many positive roots as there are changes in the signs of the terms from +

A a

to

to—, and from — to + ; and the remaining roots are negative. The rule also may be demonstrated.

Note. The impossible roots in this rule are supposed to be either positive or negative.

In this example of a numeral equation $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, the roots are, +1, +2, +3, +4, and the preceding observations with regard to the signs and coefficients take place.

Cor. If a term of an equation is wanting, the positive and negative parts of its coefficient must then be equal. If there is no absolute term, then some of the roots must be = 0, and the equation may be depressed by dividing all the terms by the lowest power of the unknown quantity in any of them. In this case also, $x - 0 = 0$, $x - 0 = 0$, &c. may be considered as so many of the component simple equations, by which the given equation being divided, it will be depressed so many degrees.

C H A P II.

Of the Transformation of Equations.

THERE are certain transformations of equations necessary towards their solution ; and the most useful are contained in the following propositions.

Prop. I. The affirmative roots of an equation become negative, and the negative become affirmative, by changing the signs of the alternate terms, beginning with the second.

Thus the roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ are $+1, +2, +3, -5$, whereas the roots of the equation, $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, are $-1, -2, -3, +5$.

The

The reason of this is derived from the composition of the coefficients of these terms, which consist of combinations of odd numbers of the roots, as explained in the preceding chapter.

Prop. II. An equation may be transformed into another that shall have its roots greater or less than the roots of the given equation by some given difference.

Let x be the unknown quantity of the equation, and e the given difference; let $y = x \pm e$, then $x = y \mp e$; and if for x and its powers in the given equation, $y \mp e$ and its powers be inserted, a new equation will arise, in which the unknown quantity is y , and its value will be $x \pm e$; that is, its roots will differ from the roots of the given equation by e .

Let the equation proposed be $x^3 - px^2 + qx - r = 0$, of which the roots must be diminished by e . By inserting for x and its powers, $y + e$ and its powers, the equation required is,

$$\left. \begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ -py^2 - 2pey - pe^2 \\ + qy + qe \\ -r \end{array} \right\} = 0.$$

Cor. 1. From this transformation, the second, or any other intermediate term, may be taken away; granting the resolution of equations.

Since the coefficients of all the terms of the transformed equation, except the first, involve the powers of e and known quantities only, by putting the coefficient of any term equal to 0, and resolving that equation, a value of e may be determined; which being substituted, will make that term to vanish.

Thus, in this example, to take away the second term, let its coefficient, $3e - p = 0$, and $e = \frac{1}{3}p$ which being substituted for e , the new equation will want the second term. And universally, the coefficient of the first term of a cubic equation being 1, and x being the unknown quantity, the second term may be taken away by supposing $x = y \mp \frac{1}{3}p, \pm p$ being the coefficient of that term.

Cor.

Cor. 2. The second term may be taken away by the solution of a simple equation, the third by the solution of a quadratic, and so on.

Cor. 3. If the second term of a quadratic equation be taken away, it will become a pure equation, and thus a solution of quadratics will be obtained, which coincides with the solution already given in Part. I.

Cor. 4. The last term of the transformed equation is the same with the given equation, only having e in place of x .

Prop. III. In like manner may an equation be transformed into another, of which the roots shall be equal to the roots of the given equation, multiplied or divided by a given quantity.

Let x be the unknown letter in the given equation, and y that of the equation wanted ; also let e be the given quantity.

To multiply the roots let $xe=y$, and $x = \frac{y}{e}$.

To divide the roots let $\frac{x}{e}=y$, and $x=ye$.

Then

Then substitute for x and its powers, $\frac{y}{a}$ or ye and its powers, and the new equation of which y is the unknown quantity, will have the property required.

Cor. 1. By this proposition an equation, in which the coefficient of the first term is any known quantity, as a , may be transformed into another, in which the coefficient of the first term shall be unit. Thus, let the equation be $ax^3 - px^2 + qx - r = 0$. Suppose $y = ax$, or $x = \frac{y}{a}$, and for x and its powers insert $\frac{y}{a}$ and its powers, and the equation becomes $\frac{y^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$, or $y^3 - py^2 + qay - a^3r = 0$. Also, let the equation be $5x^3 - 6x^2 + 7x - 30 = 0$; and if $x = \frac{y}{5}$, then $y^3 - 6y^2 + 35y - 750 = 0$.

Cor. 2. If the two transformations in Prop. 2. and 3. be both required, they may be performed either separately or together.

Thus, if it is required to transform the equation $ax^3 - px^2 + qx - r = 0$ into one which shall want the second term, and in which the coefficient of the first term shall be

be 1; let $x = \frac{y}{a}$, and then $y^3 - py^2 + qay - a^2r = 0$ as before; then let $y = z + \frac{1}{3}p$, and the new equation, of which z is the unknown quantity, will want the second term, and the coefficient of z^3 , the highest term is 1. Or, if $x = \frac{z + \frac{1}{3}p}{a}$, the same equation as the last found, will arise from one operation.

Ex. Let the equation be $5x^3 - 6x^2 + 7x - 30 = 0$. If $x = \frac{y}{5}$, then $y^3 - 6y^2 + 35y - 750 = 0$. And if $y = z + 2$, $z^3 + 23z - 696 = 0$. Also, at once, let $x = \frac{z+2}{5}$, and the equation properly reduced, by bringing all the terms to a common denominator, and then casting it off, will be $z^3 + 23z - 696 = 0$, as before.

Cor. 3. If there are fractions in an equation, they may be taken away, by multiplying the equation by the denominators, and by this Prop. the equation may then be transformed into another, without fractions, in which the coefficient of the first term is 1. In like manner, may a surd coefficient be taken away in certain cases.

Cor.

Cor. 4. Hence also, if the coefficient of the second term of a cubic equation is not divisible by 3, the fractions thence arising in the transformed equation, wanting the second term, may be taken away by the preceding corollary. But the second term also may be taken away, so that there shall be no such fractions in the transformed equation, by supposing $x = \frac{z \mp p}{3}$, $\pm p$ being the coefficient of the second term of the given equation. And, if the equation $ax^3 - px^2 + qx - r = 0$ be given, in which p is not divisible by 3, by supposing $x = \frac{z+p}{3a}$, the transformed equation reduced is $z^3 - \frac{3p^2 + 9aq}{a^2} \times z - \frac{2p^3 + 9apq - 27a^2r}{a^3} = 0$; wanting the second term, having 1 for the coefficient of the first term, and the coefficients of the other terms being all integers, the coefficients of the given equation being also supposed integers.

General Corollary to Prop. I. II. III.

If the roots of any of these transformed equations be found by any method, the roots of the

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original

original equation, from which they were derived, will easily be found from the simple equations expressing their relation. Thus, if 8 is found to be a root of the transformed equation $z^3 + 23z - 696 = 0$ (Cor. 2. prop. 3.) Since $x = \frac{z+2}{5}$, the corresponding root of the given equation $5x^3 - 6x^2 + 7x - 30 = 0$ must be $\frac{8+2}{5} = 2$. It is to be observed also, that the reasoning in Prop. 2. and 3. and the corollaries, may be extended to any order of equations, though in them it is applied chiefly to cubics.

C H A P.

C H A P III.

Of the Resolution of Equations.

FROM the preceding principles and operations, rules may be derived for resolving equations of all orders.

I. *CARDAN'S Rule for Cubic Equations.*

The second term of a cubic equation being taken away, and the coefficient of the first term being made 1, (by Cor. 1. Prop. 2. and Cor. 1. Prop. 3. Chap. II.) it may be generally represented by $x^3 + 3qx + 2r = 0$; the sign + in all terms denoting the addition of them, with their proper signs. Let $x = m + n$, and also $mn = -q$; by the substitution of these values, an equation of
the

the 6th order, but of the quadratic form, is deduced, which gives the values of m and n , and hence ;

$$(m+n) x = \sqrt[3]{-r + \sqrt{r^2 + q^3}} + \sqrt[3]{-r - \sqrt{r^2 + q^3}};$$

$$\text{or } x = \sqrt[3]{-r + \sqrt{r^2 + q^3}} - \frac{q}{\sqrt[3]{-r + \sqrt{r^2 + q^3}}}.$$

Cor. 1. In the given equation, if $3q$ is negative, and if r^2 is less than q^3 , this expression of the root involves impossible roots ; while, at the same time, all the roots of that equation are possible. The reason is, that, in this method of solution, it is necessary to suppose that x the root, may be divided into two parts, of which the product is q . But it is easy to shew, that, in this, which is called the *irreducible case*, it cannot be done.

For example, the equation (Ex. 3. Sect. 3. of this chapter), $x^3 - 156x + 560 = 0$, belongs to the irreducible case, and the three roots are $+4$, $+10$, -14 , and it is plain that none of these roots can be divided into two parts (m and n), of which the product can be equal to $(-q =) \frac{156}{3} = 52$; for the
greatest

greatest product from the division of the greatest root -14 , is $-7 \times -7 = 49$, less than 52.

If the cube root of the compound surd can be extracted, the impossible parts balance each other, and the true root is obtained.

The geometrical problem of the trisection of an arch is resolved algebraically, by a cubic equation of this form; and hence the foundation of the rule for resolving an equation belonging to this case, by a table of sines.

Cor. 2. Biquadratic equations may be reduced to cubics, and may therefore be resolved by this rule.

Some other classes of equations too, may be resolved by particular rules; but these, and every other order of equations, are commonly resolved by the general rules, which may be equally applied to all.

II. *Solution of Equations, whose Roots are commensurate.*

R U L E I.

All the terms of the equation being brought to one side, find all the divisors of the absolute term, and substitute them successively in the equation for the unknown quantity. That divisor which substituted in this manner gives the result $=0$, shall be a root of the equation.

$$\text{Ex. 1. } \left. \begin{array}{l} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ -bx^2 + 3abx \end{array} \right\} = 0.$$

The simple literal divisors of $-2a^2b$, are $a, b, 2a, 2b$, any of which may be inserted for x . Supposing $x = +a$, the equation becomes

$$\left. \begin{array}{l} a^3 - 3a^3 + 2a^3 - 2a^2b \\ -ba^2 + 3a^2b \end{array} \right\} \text{ which is obviously } = 0.$$

$$\text{Ex. 2. } x^3 - 2x^2 - 33x + 90 = 0.$$

The divisors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90.

The first of these divisors, which being inserted for x , will make the result $=0$, is
+3;

$+3$; $+5$ is another, and it is plain the last root must be negative, and it is -6 .

When 3 is discovered to be a root, the given equation may be divided by $x-3=0$, and the result will be a quadratic, which being resolved will give the other two roots, $+5$ and -6 .

The reason of the rule appears from the property of the absolute term formerly defined, viz. that it is the product of all the roots.

To avoid the inconvenience of trying many divisors, this method is shortened by the following

R U L E II.

Substitute in place of the unknown quantity successively three or more terms of the progression, 1, 0, -1 , &c. and find all the divisors of the sums that result; then take out all the arithmetical progressions that can be found among these divisors whose common difference is 1, and the values of x will

will be among those terms of the progressions which are the divisors of the result arising from the substitution of $x=0$. When the series increases, the roots will be positive; and when it decreases, the roots will be negative.

E X A M P L E.

Let it be required to find a root of the equation $x^3 - x^2 - 10x + 6 = 0$.

The operation is thus :

Supposit.	Result.	Divisors	Ar. pro.
$x=1$	$x^3 - x^2 - 10x + 6 =$	-4	$1, 2, 4$
$x=0$		$+6$	$1, 2, 3, 6$
$x=-1$		$+14$	$1, 2, 7, 14$

In this example there is only one progression, 4, 3, 2, and therefore 3 is a root, and it is -3 , since the series decreases.

It is evident from the rules for transforming equations (Chap. 2.) that by inserting for x , $+1 (= +e)$ the result is the absolute term of an equation of which the roots are less

less than the roots of the given equation by 1 ($=e$). Cor. 4. Prop. 2. When $x=0$ the result is the absolute term of the given equation. When for x is inserted -1 ($=-e$) the result is the absolute term of an equation whose roots exceed the roots of the given equation by 1 ($=e$). Hence, if the terms of the series 1, 0, $-1, -2$, &c. be inserted successively for x , the results will be the absolute terms of so many equations, of which the roots form an increasing arithmetical series with the difference 1. But, as the commensurate roots of these equations must be among the divisors of their absolute terms, hence they must be among the arithmetical progressions found by this rule. The roots of the given equation therefore are to be sought for among the terms of these progressions which are divisors of the result, upon the supposition of $x=0$, because that result is its absolute term.

It is plain that the progressions must always be increasing, only it is to be observed, that a decreasing series with the sign $+$ becomes increasing with the sign $-$. Thus, in the preceding example, $-4, -3, -2$, is

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an

an increasing series, of which -3 is to be tried, and it succeeds.

If, from the substitution of three terms of the progression, $1, 0, -1$, &c. there arise a number of arithmetical serieses, by substituting more terms of that progression, some of the serieses will break off, and, of course, fewer trials will be necessary.

III. *Examples of Questions producing the higher Equations.*

E X A M P L E I.

It is required to divide 16 l. between two persons, so that the cube of the one's share may exceed the cube of the other's by 386.

Let the greater share be x pounds,

And the less will be $16-x$;

By the question, $x^3 - \overline{16-x}^3 = 386$

And by Inv. $2x^3 - 48x^2 + 768x - 4096 = 386$

Transp. and divide $x^3 - 24x^2 + 384x - 2241 = 0$.

<i>Suppos.</i>	<i>Results.</i>	<i>Divisors.</i>
If $x = 1$;	$- 1880$	$- 1, 2, 4, 5, 8, 10, 20,$
$x = 0$;	$- 2241$	$- 1, 3, 9, 27, 81.$
$x = -1$;	$- 2650$	$- 1, 2, 5, 10, 25, 50,$
		Where

Where 8, 9, 10, differ by 1; therefore +9 is to be tried; and being inserted for x , the equation is =0. The two shares then are 9 and 7 which succeed. Since $x=9$; $x-9=0$, is one of the simple equations from which this cubic is produced; therefore
$$\frac{x^3-24x^2+384x-2241}{x-9}=x^2-15x+249=0.$$
 And the two roots of this quadratic are impossible.

E X A M P L E II.

What two numbers are those, whose product multiplied by the greater will produce 405, and their difference multiplied by the less 20?

Let the greater number be x , and the less y .

Then by quest. $\begin{cases} (xy \times x) x^2 y = 405 \\ (x-y \times y) xy - y^2 = 20 \end{cases}$

Therefore - - - $x = \frac{y^2+20}{y}$

And - - - $x^2 = \frac{y^4+40y^2+400}{y^2}$

Also - - - $x^2 = \frac{405}{y}$

Therefore $\frac{y^4+40y^2+400}{y^2} = \frac{405}{y}$

Mult. and transp. $y^4+40y^2-405y+400=0.$

This

This biquadratic resolved by divisors, gives $y=5$: and therefore $x=9$. Also
$$\frac{y^4+40y^2-405y+400}{y-5} = y^3+5y^2+65y-80=0.$$

This cubic equation has one positive incommensurate root, viz. 1.114, &c. which may be found by the rule in the next section, and two impossible. The incommensurate root $y=1.114$, &c. gives $x=19.067$, &c. and these two answer the conditions very nearly.

E X A M P L E III.

The sum of the squares of two numbers 208, and the sum of their cubes 2240 being given, to find them.

Let the greater be $x+y$, and the less $x-y$.
Then $(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2 = 208$

$$\text{Hence } y^2 = 104 - x^2$$

$$\text{Also } (x+y)^3 + (x-y)^3 = 2x^3 + 6xy^2 = 2240$$

Substitute for y^2 its value and $2x^3 + 624x - 6x^3 = 2240$.

This

This reduced gives $x^3 - 156x + 560 = 0$.

The roots of this equation are $+10$, $+4$, -14 . If $x=10$ then $y=2$, and the numbers sought are 12 and 8, which give the only just solution. If $x=4$, then $y^2=88$ and $y=\sqrt{88}$. The numbers sought are therefore $4+\sqrt{88}$ and $4-\sqrt{88}$. The last is negative, but they answer the conditions. Lastly, if $x=-14$, then $y^2=-92$, hence $y=\sqrt{-92}$, is impossible; but still the two numbers $-14+\sqrt{-92}$, $-14-\sqrt{-92}$, being inserted, would answer the conditions. But it has been frequently observed, that such solutions are both useless, and without meaning.

IV. *Solution of Equations by Approximation.*

By the former rules, the roots of equations when they are commensurate may be obtained: these however more rarely occur; and when they are incommensurate, we can find only an approximate value of them, but to any degree of exactness required. There are various rules for this purpose;
one

one of the most simple, is that of Sir Isaac Newton, which shall now be explained.

LEMMA. If any two numbers being inserted for the unknown quantity (x) in any equation, give results with opposite signs, an odd number of roots must be between these numbers.

This appears from the property of the absolute term, and from this obvious maxim, that, if a number of quantities be multiplied together, and if the signs of an odd number of them be changed, the sign of the product is changed. For, when a positive quantity is inserted for x , the result is the absolute term of an equation whose roots are less than the roots of the given equation by that quantity. (Prop. 2. Cor. 3. Chap. 2.) If the result has the same sign as the given absolute term, then from the property of the absolute term (Prop. 2. Chap. 1.) either none or an even number only of the positive roots, have had their signs changed by the transformation; but if the result has an opposite sign to that of the

the given absolute term, the signs of an odd number of the positive roots must have been changed. In the first case, then, the quantity substituted must have been either greater than each of an even number of the positive roots of the given equation, or less than any of them; in the second case, it must have been greater than each of an odd number of the positive roots. An odd number of the positive roots therefore must lie between them, when they give results with opposite signs. The same observation is to be extended to the substitution of negative quantities and the negative roots.

From this lemma, by means of trials, it will not be difficult to find the nearest integer to a root of a given numeral equation. This is the first step towards the approximation; and both the manner of continuing it, and the reason of the operation, will be evident from the following example.

Let the equation be $x^3 - 2x - 5 = 0$.

1. Find the nearest integer to the root; in this case a root is between 2 and 3, for these

these numbers being inserted for x , the one gives a positive, and the other a negative result. Either the number above the root, or that below it, may be assumed as the first value; only it will be more convenient to take that which appears to be nearest to the root, as will be manifest from the nature of the operation.

2. Suppose $x=2+f$, and substitute this value of x in the equation.

$$\begin{array}{r}
 x^3 = 8 + 12f + 6ff + f^3 \\
 -2x = -4 - 2f \\
 -5 = -5 \\
 \hline
 x^3 - 2x - f = -1 + 10f + 6f^2 + f^3 = 0.
 \end{array}$$

As f is less than unit, its powers f^2 and f^3 may be neglected in this first approximation, and $10f = 1$, or $f = 0.1$ nearly, therefore $x = 2.1$ nearly.

3. As $f = 0.1$, nearly, let $f = .1 + g$, and insert this value of f in the preceding equation.

$$\begin{array}{r}
 f^3 = 0.001 + 0.03g + 0.3g^2 + g^3 \\
 6f^2 = 0.06 + 1.2g + 6g^2 \\
 10f = 1 + 10g \\
 -1 = -1 \\
 \hline
 f^3 + 6f^2 + 10f - 1 = 0.061 + 11.23g + 6.3g^2 + g^3 = 0
 \end{array}$$

and

and neglecting g^2 and g^3 as very small $0.61 + 11.23g = 0$, or $g = \frac{-0.061}{11.23} = -.0054$, hence $f = 0.1 + g = .0946$ nearly, and $x = 2.0946$ nearly.

4. This operation may be continued to any length, as by supposing $g = -.0054 + b$ and so on, and the value of $x = 2.09455147$ nearly.

By the first operation a nearer value of x may be found thus; since $f = .1$ nearly and $-1 + 10f + 6f^2 + f^3 = 0$,

$f = \frac{1}{10 + 6f + f^2}$, that is, $f = \frac{1}{10 + .6 + .01} = .094$ true to the last figure, and $x = 2.094$.

In the same manner may the root of a pure equation be found, and this gives an easy method of approximating to the roots of numbers which are not perfect powers.

This rule is applicable to numeral equations of every order, and, by assuming a general equation, general rules may be deduced for approximating to the roots of any proposed equation. By a similar method we may approximate to the roots of literal equations, which will be expressed by infinite series.

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and neglecting ϵ and ϵ' as very small, 0.01
 and 0.02, a 1st approximation for ϵ and ϵ'
 will be $\epsilon = 0.005$ and $\epsilon' = 0.01$.
 Hence $\epsilon = 0.005$ and $\epsilon' = 0.01$ and
 a 2nd approximation for ϵ and ϵ' will be
 $\epsilon = 0.004$ and $\epsilon' = 0.008$.
 This operation may be continued to
 any length by supposing $\epsilon = 0.004$ and
 $\epsilon' = 0.008$ and the value of ϵ and ϵ'
 nearly.

By the last operation a near value of ϵ
 may be found; since $\epsilon = 1$ nearly and

$$\epsilon = \frac{1 + 10\sqrt{10}}{10 + \sqrt{10}} = 0.004$$

true to the last figure, and $\epsilon' = 0.004$
 in the same manner may the root of a pure
 equation be found, and this gives an easy
 method of solving in simple powers of
 numbers which are not perfect powers.

This rule is applicable to the general equa-
 tion of every order, and, by assuming a
 proper value for ϵ and ϵ' , may be de-
 duced for approximating to the roots of a
 polynomial equation. By a similar me-
 thod we may approximate to the roots of
 literal equations which will be expedient
 by infinite series.

$$x^3 - 2x^2 + 3x - 4 = 0 \quad \text{has}$$

E L E M E N T S
O F
A L G E B R A.

P A R T III.

Of the Application of Algebra to Geometry.

C H A P. I.

General Principles.

GEOMETRY treats both of the magnitude and position of extension, and their connections.

Algebra treats only of magnitude. Therefore, of the relations which subsist in geometrical figures, those of magnitude only can be immediately expressed by algebra.

The

The opposite position of straight lines may indeed be expressed simply by the signs + and —. But, in order to express the various other positions of geometrical figures by algebra, from the principles of geometry, some relations of magnitude must be found, which depend upon these positions, and which can be exhibited by equations: And conversely, by the same principles may the positions of figures be inferred from the equations denoting such relations of their parts.

Though this application of algebra appears to be indirect, yet such is the simplicity of the operations, and the general nature of its theorems, that investigations, especially in the higher parts of geometry, are generally easier and more expeditious by the algebraical method, though less elegant than by what is purely geometrical. The connections also, and analogies of the two sciences established by this application, have given rise to many curious speculations; geometry has been rendered far more extensive and useful, and algebra itself has received considerable improvements.

I. *Of*

Of the Algebraical Expression of Geometrical Magnitudes.

A *line*, whether known or unknown, is represented by a single letter; a *rectangle* is properly expressed by the product of the two letters representing its sides; and a *rectangular parallelopiped* by the product of three letters, two of which represent the sides of any of its rectangular bases, and the third the altitude.

These are the most simple expressions of geometrical magnitudes, and any other which has a known proportion to these, may, in like manner, be expressed algebraically. Conversely, the geometrical magnitudes, represented by such algebraical quantities, may be found, only the algebraical dimensions above the third, not having any corresponding geometrical dimensions, must be expressed by proportionals *.

The

* All algebraical dimensions above the third, must be expressed by inferior geometrical dimensions; and, tho' any algebraical quantities, of two and three dimensions, may be immediately expressed by surfaces and solids respectively, yet it is generally necessary to express them, and all superior dimensions, by lines.

If

The opposite position of straight lines, it has been remarked, may be expressed by the signs $+$ and $-$.

Thus,

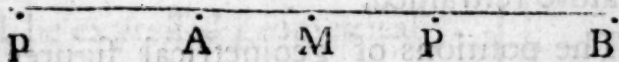
If, in any geometrical investigation by algebra, each line is expressed by a single letter, and each surface or solid by an algebraical quantity of two or three dimensions respectively, then whatever legitimate operations are performed with regard to them, the terms in any equation derived will, when properly reduced, be all of the same dimension; and any such equation may be easily expressed geometrically by means of proportionals, as in the following example.

Thus, if the algebraical equation $a^4 + b^4 = c^4 - d^4$, is to be expressed geometrically, a, b, c , and d , being supposed to represent straight lines; let $a : b : c : f : g$, in continued proportion, then $a^4 : b^4 :: a : g$ and $a^4 : a^4 + b^4 :: a : a + g$; then let $a : c : b : k : l$, and $a^4 : c^4 :: a : l$; also, let $c : d : m : n : p$, and $c^4 : d^4 :: c : p$, or $c^4 : c^4 - d^4 :: c : c - p$. By combining the two former proportions (Chap. 2, Part 1.) $c^4 : a^4 + b^4 :: l : a + g$, and combining the latter with this last found, $c^4 - d^4 : a^4 + b^4 :: c - p \times l : c \times a + g$; therefore $c - p \times l = c \times a + g$, and $c : c - p :: l : a + g$.

If any known line is assumed as 1, as its powers do not appear, the terms of an equation, including any of them, may be of very different dimensions; and before it can be properly expressed by geometrical magnitudes, the deficient dimensions must be supplied by powers of the 1. When an equation has been derived from geometrical relations, the line denoting 1 is known; and when an assumed equation is to be expressed by the relations of geometrical magnitudes, the 1 is to be assumed.

In

Thus, let a point A be given in the line



AP, any segment AP taken to the right hand, being considered as positive, a segment Ap to the left is properly represented by a negative quantity. If a and b represent two lines; and if, upon the line AB from the point A, AP be taken towards the right equal to a , it may be expressed by $+a$; then PM taken to the left and equal to b , will be properly represented by $-b$, for AM is equal to $a-b$: If $a=b$, then M will fall upon A, and $a-b=0$: By the same notation, if b is greater than a , M will fall to the left of A; and in this case, if $2a=b$, and if Pp be taken equal to b , then $(a-b=) -a$ will represent Ap, which is equal to a , and situated to the left of A. This use of the signs, however, in particular

In this manner may any single power be expressed by a line. If it is x^5 , then to 1, x find four quantities in continued proportion, so that $1:x:m:n:p:q$, then $1:q::1^5:x^5$, or $q=x^5$, and so of others.

lar cases, may be precluded, or in some measure restrained.

The positions of geometrical figures are so various, that it is impossible to give general rules for the algebraical expression of them. The following are a few examples.

An angle is expressed by the ratio of its sine to the radius; a right angle in a triangle, by putting the squares of the two sides equal to the square of the hypotenuse; the position of points is ascertained by the perpendiculars from them on lines given in position; the position of lines by the angles which they make with given lines, or by the perpendiculars on them from given points: The similarity of triangles by the proportionality of their sides which gives an equation, &c.

These and other geometrical principles must be employed both in the demonstration of theorems, and in the solution of problems. The geometrical proposition must first be expressed in the algebraical manner,

manner, and the result after the operation must be expressed geometrically.

II. *The Demonstration of Theorems.*

All propositions in which the proportions of magnitudes only are employed, also all propositions expressing the relations of the segments of a straight line, of their squares, rectangles, cubes, and parallelepipeds, are demonstrated algebraically with great ease: Such demonstrations, indeed, may in general be considered as an abridged notation of what are purely geometrical.

This is particularly the case in those propositions, which may be geometrically deduced without any construction of the squares, rectangles, &c. to which they refer. From the First proposition of the second Book of Euclid, the nine following may be easily derived in this manner, and they may be considered as proper examples of this most obvious application of algebra to geometry.

If certain positions are either supposed or to be inferred in a theorem, we must find,

E e

according

according to the preceding observations, the connection between these positions and such relations of magnitude as can be expressed and reasoned upon by algebra. The algebraical demonstrations of the 12th and 13th propositions of the 2d Book of Euclid, require only the 47th of the I. El. The 35th and 36th of the 3d. Book require only the 3. III. El. and 47. I. El.

From few simple geometrical principles alone, a number of conclusions, with regard to figures, may be deduced by algebra; and to this in a great measure is owing the extensive use of this science in geometry. If other more remote geometrical principles are occasionally introduced, the algebraical calculations may be much abridged. The same is to be observed in the solution of problems; but such in general are less obvious, and more properly belong to the strict geometrical method.

III. *Of the Solution of Problems.*

Upon the same principles are geometrical problems to be resolved. The problem
is

is supposed to be constructed, and proper algebraical notations of the known and unknown magnitudes are to be sought for, by means of which their connections may be expressed by equations. It may first be remarked, as was done in the case of theorems, that in those problems which relate to the divisions of a line, and the proportions of its parts, the expression of the quantities, and the stating their relations by equations, are so easy as not to require any particular directions. But, when various positions of geometrical figures, and their properties are introduced, the solution requires more attention and skill. No general rules can be given on this subject, but the following observations may be of use.

1. The construction of the problem being supposed, it is often farther necessary to produce some of the lines till they meet ; to draw new lines joining remarkable points ; to draw lines from such points perpendicular or parallel to other lines, and such other operations as seem conducive to the finding of equations ; and for this purpose, those especially

especially are to be employed, which divide the scheme into triangles that are given, right angled or similar.

2. It is often convenient to denote by letters, not the quantities particularly sought, but some others from which they can easily be deduced. The same may be observed of given quantities.

3. The proper notation being made, the necessary equations are to be derived by the use of the most simple geometrical principles, such as the addition and subtraction of lines or of squares, the proportionality of lines, particularly of the sides of similar triangles, &c.

4. There must be as many independent equations as there are unknown quantities assumed in the investigation, and from these a final equation may be inferred by the rules of PART I.

If the final equation from the problem be resolved, the roots may often be exhibited

ted geometrically; but the geometrical construction of problems may be effected also, without resolving the equation, and even without deducing a final equation, by the methods afterwards to be explained.

If the final equation is simple or quadratic, the roots being obtained by the common rules, may be geometrically exhibited by the finding of proportionals, and the addition or subtraction of squares.

By inserting numbers for the known quantities, a numeral expression of the quantities sought will be obtained by resolving the equation. But in order to determine some particulars of the problem, besides finding the unknown quantities of the equation, it may be farther necessary to make a simple construction; or, if it is required that every thing be expressed in numbers, to substitute a new calculation in place of that construction.

P R O P.

P R O P. I.

To divide a given straight line AB into two parts, so that the rectangle contained by the whole line, and one of the parts, may be equal to the square of the other part. This is prop. 11th II. B. of Eucl.

$\overline{\text{C} \qquad \qquad \qquad \text{A} \qquad \text{C} \qquad \text{B}}$

Let C be the point of division, and let $AB=a$, $AC=x$, and then $CB=a-x$. From the problem $a^2-ax=x^2$; and this equation being resolved (Chap. V. P. II.) gives $x=\pm \sqrt{\frac{a^2+a^2}{4}}-\frac{a}{2}$.

The quantity $\sqrt{\frac{a^2+a^2}{4}}$, is the hypothenuse of a right angled triangle of which the two sides are a , and $\frac{a}{2}$, and is therefore easily found; $\frac{a}{2}$ being taken from this line, gives $x=AC$, which is the proper solution. But if a line AC be taken on the opposite side

side of A, and equal to the above mentioned hypotenuse, together with $\frac{a}{2}$, it will represent the negative root $-\sqrt{a^2 + \frac{a^2}{4}} - \frac{a}{2}$, and will give another solution; for in this case also $AB \times Bc = Ac^2$. But c is without the line AB, and therefore if it is not considered as making a division of AB, this negative root is rejected.

This solution coincides with what is given by Euclid. For $\sqrt{a^2 + \frac{a^2}{4}}$ is equal (see the fig. of Prop. 11th II. B. Eucl. Simson's edit.) to EB or EF, and therefore $x = \sqrt{a^2 + \frac{a^2}{4}} - \frac{a}{2} = EF - EA = AF = AH$; and the point H corresponds to C in the preceding figure.

Besides, if on $(EF + EA =) CF$ (instead of $EF - EA = FA$) a square be described on the opposite side of CF from AG, BA produced will meet a side of it in a point, which if it be called K, will give $KB \times BA = KA^2$. K corresponds to c , and this solution will correspond with the algebraical solution, by means of the negative root.

If CB had been called x , and $AC = a - x$, the equation would be $ax = a^2 - 2ax + x^2$,
which

which gives $x = \frac{3a \pm \sqrt{5a^2}}{2}$, in which both roots are positive, and the solutions derived from them coincide with the preceding. If the solution be confined to a point within the line, then one of these positive roots must be rejected, for one of the roots of the compound square from which it is derived,

$x - \frac{3a}{2}$, a negative quantity, which in this strict hypothesis, is not admitted. In such a problem, however, both constructions are generally received, and considered even as necessary to a complete solution of it.

If a solution in numbers be required, let $AB = 10$, then $x = \pm \sqrt{125} - 5$. It is plain, whatever be the value of AB , the roots of this equation are incommensurate, though they may be found, by approximation, to any degree of exactness required. In this case $x = \pm 11.1803 - 5$, nearly, that is $AC = 6.1803$ nearly; and $Ac = 16.1803$ nearly.

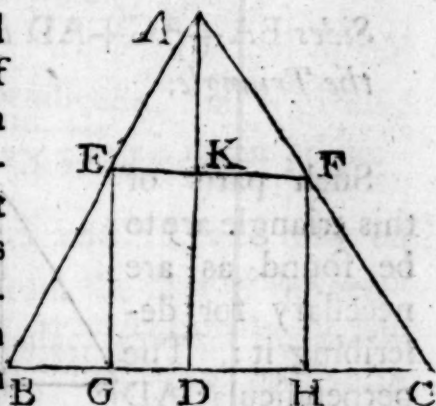
P R O B.

P R O B. II.

In a given Triangle ABC to inscribe a Square.

Suppose it to be done, and let it be EFHG; from A let AD be perpendicular on the base BC, meeting EF in K.

Let $BC = a$, and $AD = p$, both of which are given because the triangle is given: Let AK be assumed as the unknown quantity, because from it the square can easily be constructed, and let it be called x : Then ($KD = EG$ $\Rightarrow$) $EF = p - x$.



On account of the parallels EF, BC, $AD : BC :: AK : EF$; that is, $p : a :: x : p - x$, and $p^2 - px = ax$, which equation being resolved, gives $x = \frac{p^2}{p+a}$.

Therefore x or AK is a third proportional to $p+a$, and p , and may be found by 11.

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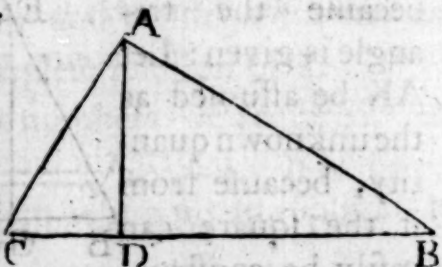
VI.

VI. El. The point K being found, the construction of the square is sufficiently obvious.

P R O B. III.

In the right-angled Triangle ABC, the Base BC and the Sum of the Perpendicular and Sides BA+AC+AD being given, to find the Triangle.

Such parts of this triangle are to be found as are necessary for describing it: The perpendicular AD



will be sufficient for this purpose, and let it be called x : Let $AB+AC+AD=a$, $BC=b$, therefore $BA+AC=a-x$: Let $BA-AC$ be denoted by y , then $BA=\frac{a+y-x}{2}$,

and $AC=\frac{a-x-y}{2}$. But [47. I. El.] $BC^2=$

BA^2+AC^2 which being expressed algebrai-

cally, becomes, $b^2=\left(\frac{a+y-x}{2}\right)^2+\left(\frac{a-x-y}{2}\right)^2=$

$\frac{a^2-2ax+x^2+y^2}{2}$. Likewise, from a known

property

property of right angled triangles, $BC \times AD = BA \times AC$; that is $bx = \left(\frac{a+y-x}{2} \times \frac{a-y-x}{2} \right) = \frac{a^2 - 2ax + x^2 - y^2}{4}$. This last equation being

multiplied by 2, and added to the former, gives $b^2 + 2bx = a^2 - 2ax + x^2$, which being resolved according to the rules of Part I. Chap. 5. gives $x = a + b - \sqrt{2ab + 2b^2}$.

To construct this: $a + b$ is the sum of the perimeter and perpendicular, and is given; $\sqrt{2ab + 2b^2} = \sqrt{a+b \times 2b}$ is a mean proportional between $a + b$ and $2b$, and may be found; therefore, from the sum of the perimeter and perpendicular, subtract the mean proportional between the said sum and double the base, and the remainder will be the perpendicular required.

From the base and perpendicular, the right-angled triangle is easily constructed.

In numbers, let $BA + AC + AD = 18.8 = a$; $BC = 10 = b$; then $AD = a + b - \sqrt{2ab + b^2} = 28.8 - \sqrt{576} = 4.8 = x$, and $BA + AC = 14$. By either of the first equations $y^2 = 2b^2 + 2ax - a^2 - x^2 = 4$ and $y = BA - AC = 2$; therefore $BA = 8$, and $AC = 6$.

The

The geometrical expression of the roots of final equations arising from problems, may be found without resolving them, by the intersection of geometrical lines. Thus, the roots of a quadratic are found by the intersections of the circle and straight line, those of a cubic and biquadratic, by the intersections of two conic sections, &c.

The solution of problems may be effected also by the intersections of the *loci* of two intermediate equations without deducing a final equation: But these two last methods can only be understood by the doctrine of the *loci* of equations.

C H A P. II.

Of the Definition of Lines by Equations.

LINES which can be mathematically treated of, must be produced according to an uniform rule, which determines the position of every point of them.

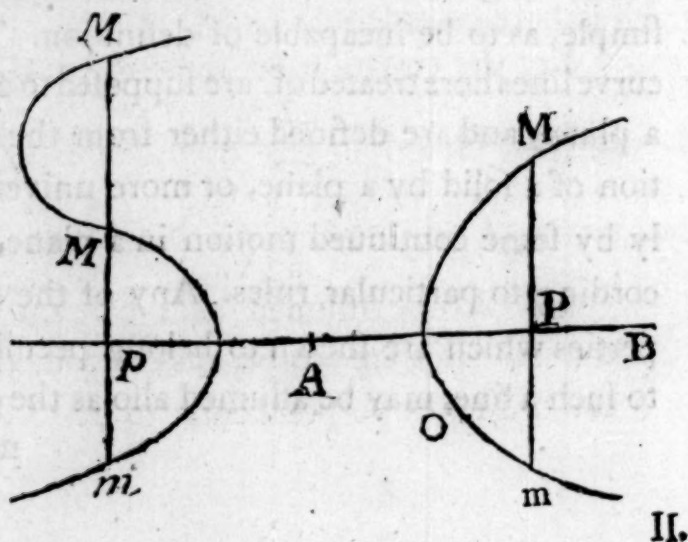
This rule constitutes the *definition* of any line from which all its other properties are to be derived.

A straight line has been considered as so simple, as to be incapable of definition. The curve lines here treated of, are supposed to be in a plane, and are defined either from the section of a solid by a plane, or more universally by some continued motion in a plane, according to particular rules. Any of the properties which are shewn to belong peculiarly to such a line, may be assumed also as the definition

nition of it, from which all the others, and even what upon other occasions may have been considered as the primary definition, may be demonstrated. Hence lines may be defined in various methods, of which the most convenient is to be determined by the purpose in view. The simplicity of a definition, and the ease with which the other properties can be derived from it, generally give a preference.

D E F I N I T I O N S.

I. When curve lines are defined by equations, they are supposed to be produced by the extremity of one straight line, as PM moving in a given angle along another straight line AB given in position which is called the base.



II. The straight line PM moving along the other, is called an *Ordinate*, and is usually denoted by y .

III. The segment of the base AP between a given point in it A, and an ordinate PM, is called an *Absciss* with respect to that ordinate, and is denoted by x . The ordinate and absciss together are called *Co-ordinates*.

IV. If the relation of the variable absciss and ordinate AP and PM, be expressed by an equation, which besides x and y contains only known quantities, the curve MO described by the extremity of the ordinate, moving along the base, is called the *Locus* of that equation.

V. If the equation is finite, the curve is called *Algebraical* *. It is this class only which is here considered.

VI.

* The terms *Geometrical* and *Algebraical*, as applied to curve lines, are used in different senses, by different writers; there are several other classes of curves besides what is here called algebraical, which can be treated of mathematically, and even by means of algebra. See Scholium at the end.

VI. The *dimensions* of such equations are estimated from the highest sum of the exponents of x and y in any term.—According to this definition, the terms x^4 , x^3y , x^2y^2 , xy^3 , y^4 are all of the same dimension.

VII. Curve lines are divided into *orders* from the dimensions of their equations, when freed from fractions and surds.

In these general definitions, the straight line is supposed to be comprehended, as it is the locus of simple equations. The loci of quadratic equations are shewn to be the conic sections, which are hence called lines of the second order, &c.

It is sufficiently plain from the nature of an equation, containing two variable quantities, that it must determine the position of every point of the curve, defined by it in the manner now described: for if any particular known value of one of the variable quantities as x be assumed, the equation will then have one unknown quantity only, and being resolved, will give a precise number of

of corresponding values of y , which determine so many points of the curve.

As every point of the locus of an equation has the same general property, it must be one curve only, and from this equation all its properties may be derived. It is plain also, that any curve line defined from the motion of a point, according to a fixed rule, must either return into itself, or be extended *ad infinitum* with a continued curvature.

The equation, however, is supposed to be irreducible; because, if it is not, the locus will be a combination of inferior lines: but this combination will possess the general properties of the lines of the order of the given equation.

It is to be observed all along, that the positive values of the ordinate, as PM, being taken upwards, the negative Pm will be placed downwards, on the opposite side of the base: and if positive values of the absciss, as AP, be assumed to the right from its beginning, the negative values, AP will be

G g

upon

upon the left, and from these the points of the curve M, m , on that side are to be determined.

In the general definition of curves it is usual to suppose the co-ordinates to be at right angles. If the locus of any equation be described, and if the absciss be assumed on another base, and the ordinate be placed at a different angle, the new equation expressing their relation, though of a different form, will be of the same order as the original equation ; and likewise will have, in common with it, those properties which distinguish the equations of that particular curve.

This method of defining curves by equations may not be the fittest for a full investigation of the properties of a particular curve ; but, as their number is without limit, such a minute inquiry concerning all, would be not only useless, but impossible. It has this great advantage, however, that many of the general affections of all curves, and of the distinct orders, and also some of the

the most useful properties of particular curves, may be easily derived from it.

I. The Determination of the Figure of a Curve from its Equation.

The general figure of the curve may be found by substituting successively particular values of x the absciss, and finding by the resolution of these equations the corresponding values of y the ordinate, and of consequence so many points of the curve. If numeral values be substituted for x , and also certain numbers for the known letters, the resolution of the equation gives numeral expressions of the ordinates; and from these, by means of scales, a mechanical description of the curve will be obtained, which may often be useful, both in pointing out the general disposition of the figure, and also in the practical applications of geometry.

Some more general suppositions may be of use in determining the figure; but these
can

can be suggested only from the particular form of the equation in view. By supposing x to have certain relations to the known quantities, the values of y may become more simple, and the equation may be reduced to such a form as to shew the direction of the curve, and some of its obvious properties.

The following general observations may also be laid down.

1. If in any case a value of y vanishes, then the curve meets the base in a point determined by the corresponding value of x . Hence by putting $y=0$, the roots of the equation, which in that situation are values of x , will give the distances on the base from the point assumed as the beginning of x , at which the curve meets it.
2. If at a particular value of x , y becomes infinite, the curve has an infinite arc, and the ordinate at that point becomes an asymptote.
3. If when x becomes infinitely great, y vanishes, the base becomes an asymptote.

4. If any value of y becomes impossible, then so many intersections of the ordinate and curve vanish. If at any value of x all the values of y become impossible, the ordinate does not there meet the curve.

5. If two values of y become equal and have the same sign, the ordinate in that situation either touches the curve, or passes through an intersection of two of its branches, which is called a *punctum duplex*, or through an oval become infinitely little, called a *punctum conjugatum*.

In like manner is a *punctum triplex*, &c. to be determined.

The following example will illustrate this doctrine.

Let the equation be $ay^2 - xy^2 = x^3 + bx^2$:

Therefore, $y^2 = \frac{x^3 + bx^2}{a - x}$ and $y = \pm \sqrt{\frac{x^3 + bx^2}{a - x}}$

$$= \pm \sqrt{\frac{x+b}{a-x}} \times x.$$

Let AB be assumed as a base on which the abscissas are to be taken from A, and the ordinates perpendicular to it.

Since

Since the two values of y are equal, but have opposite signs; PM , and Pm which represent them, must be taken equal to each other on opposite sides of AB ; and it is plain that the parts of the curve on the two sides of AB , must be every way similar and equal.

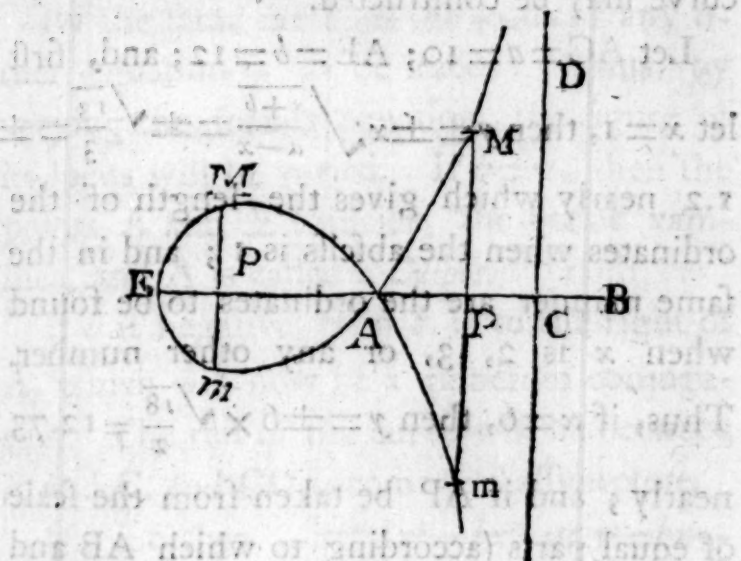
If x is made equal to a , then $y = x \sqrt{\frac{x+b}{0}}$

which is an algebraical expression for infinity; therefore if AC is taken equal to a , the perpendicular CD will become an asymptote to the curve, which will have two infinite arcs (Obs. 2.). If x is greater than a , the quantity under the radical sign becomes negative, and the values of y are impossible; that is, no part of the curve lies beyond CD .

(4.)

Both branches of the curve pass through A , since $y=0$, when $x=0$. (1.) Let x be negative, and $y = \pm x \sqrt{\frac{b-x}{a+x}}$; the values of y will be possible, if x is not greater than b ; but if $x=b$, then $y=0$, and if x is greater than b , the values of y become impossible; that is, if the absciss AP be taken to the left of A ,
and

and less than b , there will be two real equal values of y , viz. PM , Pm on the opposite sides as before; if AE be taken equal to b the curve will pass through E , and no part of it is beyond E . (1. and 4.)



The portion between A and E is called a *Nodus*.

If y be put $= 0$, then the values of x are $0, 0, -b$. That is, the curve passes twice through A , or A is a punctum duplex, and it passes also through E as before. (1.)

The mechanical description of curves mentioned in the beginning of this section, may be illustrated by the preceding example.

ample. For this purpose, let any numeral values of a and b be assumed ; and if successive numeral values of x be inserted, corresponding numeral values of y will be obtained, by which so many points in the curve may be constructed.

Let $AC=a=10$; $AE=b=12$; and, first let $x=1$, then $y=\pm x\sqrt{\frac{x+b}{a-x}}=\pm\sqrt{\frac{13}{3}}=\pm$

1.2 nearly which gives the length of the ordinates when the absciss is 1 ; and in the same manner are the ordinates to be found when x is 2, 3, or any other number.

Thus, if $x=6$, then $y=\pm 6\times\sqrt{\frac{18}{2}}=12.73$

nearly ; and if AP be taken from the scale of equal parts (according to which AB and AE are supposed to be laid down) and equal to 6, then PM, Pm , being taken from the same scale, each equal to 12.73, will give the points of the curve M, m . In

like manner, If $x=-y, y=\pm 9\sqrt{\frac{3}{19}}=\pm 3.58$, nearly ; and if $AP=9$, then PM, Pm being taken from the same scale equal to 3.58, will give the points M, m . In the

same
may be illustrated by the preceding ex-
ample

same manner may any number of points be found, and these being joined, will give a representation of the curve, which will be more or less just, according to the number of points found, and the accuracy of the several operations employed.

By the same methods the locus of any other equation is to be traced: Thus, by varying the former equation, the figure of its locus will be varied. If $b=0$, then the points A and E coincide, the nodus vanishes, and A is called a *Cuspis*.

If b is negative, then E is to the right of A, which will now be a punctum conjugatum. The rest of the curve will be between E and C, and CD becomes an asymptote.

If $a=0$ then $-xy^2=x^3-bx^2$ or $y^2=bx-x^2$ which is an equation to the circle of which $b=AE$ is the diameter.

II. *General Properties of Curves from their Equations.*

The general properties of equations lead to the general affections of curve lines. For example,

H h

A

A straight line may meet a curve in as many points as there are units in the dimension of its equation ; for so many roots may that equation have. An asymptote may cut a curve line in as many points excepting two, as it has dimensions, and no more. The same may be observed of the tangent.

Impossible roots enter an equation by pairs, therefore the intersections of the ordinate and curve must vanish by pairs.

The curves of which the number expressing the order is odd, must have at least two infinite arcs ; for the abscissas may be so assumed, that, for every value of it, either positive or negative, there must be at least one value of y , &c.

The properties of the coefficients of the terms of equations mentioned Part II. Chap. I. furnish a great number of the curious and universal properties of curve lines. For example, the second term of an equation is the sum of the roots with the signs changed, and if the second term is wanting, the positive and negative roots must be equal. From this it is easy to demonstrate, " That,
if

if each of two parallel straight lines meet a curve line in as many points as it has dimensions, and if a straight line cut these two parallels, so that the sum of the segments of each on one side, be equal to the sum of the segments on the other, this straight line will cut any other line parallel to these in the same manner." Analogous properties, with many other consequences from them, may be deduced from the composition of the coefficients of the other terms.

Many properties of a particular order of curves may be inferred from the properties of equations of that order. Thus, "if a straight line cut a curve of the third order in three points, and if another straight line be drawn, making a given angle with the former, and cutting the curve also in three points, the parallelopiped by the segments of one of these lines between its intersection with the other, and the points where it meets the curve, will be to the parallelopiped by the like segments of the other line in a given ratio." This depends upon the composition of the absolute term, and may be extended to curves of any order.

III.

III. *The Subdivision of Curves.*

As lines are divided into orders from the dimensions of their equations, in like manner, from the varieties of the equations of any order, may different *Genera* and *Species* of that order be distinguished, and from the peculiar properties of these varieties, may the affections of the particular curves be discovered.

For this purpose a complete general equation is assumed of that order, and all the varieties in the terms and coefficients which can affect the figure of the locus are enumerated.

It was formerly observed, that the equations belonging to any one curve, may be of various forms, according to the position of the base, and the angle which the ordinate makes with it, though they be all of the same order, and have also certain properties, which distinguish them from the other equations of that order.

The locus of simple equations is a *straight line*. There are *three* species of lines of the second order, which are easily shewn to be the *conic sections*, reckoning the circle and ellipse

ellipse to be one. *Seventy-eight* species have been numbered of the third order : And, as the superior orders become too numerous to be particularly reckoned, it is usual only to divide them into certain general classes.

A complete arrangement of the curves of any order, would furnish canons, by which the species of a curve whose equation is of that order might be found.

IV. *Of the place of Curves defined from other principles, in the Algebraical System.*

If a curve line be defined from the section of a solid, or from any rule different from what has been here supposed, an equation to it may be derived, by which its order and species in the *algebraical* system may be found. And, for this purpose, any base and any angle of the co-ordinates may be assumed, from which the equation may
be

be most easily derived, or may be of the most simple form.

The three *Conic Sections* are of the second order, as their equations are universally quadratic; the *Cissoïd* of the ancients is of the third order, and the forty-second species, according to Sir Isaac Newton's enumeration; this is the curve defined by the equation in page 241. when $b=0$. The curve delineated in page 239. is the 41st species. When b is negative in that equation, the locus is the 43d species. The *Conchoid* of *Nicomedes* is of the fourth order; the *Cassian* curve is also of the fourth order, &c.

It is to be observed, that not only the first definition of a curve may be expressed by an equation, but likewise any of those theorems called *loci*, in which some property is demonstrated to belong to every point of the curve. The expression of these propositions by equations, is sometimes difficult; no general rules can be given; and it must be left to the skill and experience of the learner.

SCHOLIUM.

S C H O L I U M.

This method of treating curve lines by equations, besides the uses already hinted at, has many others, which do not belong to this place ; such are, the finding the tangents of curves, their curvature, their areas and lengths, &c. The solution of these problems has been accomplished by means of the equations to curves, though by employing, concerning them a method of reasoning different from what has been here explained.

CHAP.

and let θ be the angle which the line OP makes with the horizontal axis. Then the coordinates of P are $(r \cos \theta, r \sin \theta)$. The polar coordinates of P are (r, θ) . The polar coordinates of a point P are (r, θ) if P is at a distance r from the origin O and the line OP makes an angle θ with the horizontal axis. The polar coordinates of a point P are (r, θ) if P is at a distance r from the origin O and the line OP makes an angle θ with the horizontal axis.

The method of finding the area of a region bounded by a curve in polar coordinates is as follows. Let $r = f(\theta)$ be the equation of the curve in polar coordinates. Let θ_1 and θ_2 be the values of θ for which the curve intersects the horizontal axis. Then the area of the region bounded by the curve and the horizontal axis is given by the formula
$$A = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta$$

The area of a region bounded by a curve in polar coordinates can be found by using the formula
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 where $r = f(\theta)$ is the equation of the curve in polar coordinates and θ_1 and θ_2 are the values of θ for which the curve intersects the horizontal axis. The area of a region bounded by a curve in polar coordinates can be found by using the formula
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C H A P. III.

I. *Construction of the Loci of Equations.*

THE description of a curve, according to the definition of it, is assumed in geometry as a *Postulate* *.

I i

If

* A postulate in geometry seems to be improperly called a mechanical principle. No geometrical line whatever, not even the straight line or circle, can be described mechanically according to the mathematical definition; and therefore the solutions of problems by the conic sections, or by any of the higher orders, is to be considered, in theory, as equally perfect with those by the circle and straight lines. It is a rule in strict geometry, not to employ a curve line in the solution of a problem, if it can be performed by means of a line of an inferior order; but, when a practical solution is required, then those lines, of whatever order, or of whatever class; and those methods of describing them

If the properties of a particular curve are investigated, it will appear that it may be described from a variety of data different from those assumed in the postulate, by demonstrating the dependence of the former upon the latter.

As the definitions of a curve may be various, so also may be the postulates, and a definition is frequently chosen from the mode of description connected with it. The particular object in view, it was formerly remarked, must determine the proper choice of

them are to be preferred, by which the construction required may be most easily and accurately performed. Thus, even the 2d and 3d prop. of I. B. of Euclid are constructed in practice, with much more ease and accuracy, by transferring a distance in a pair of compasses, than by the methods there described; but that principle not being assumed by Euclid as a postulate, could not be admitted in the construction of any problem in his elements. There are but few mechanical operations which admit of tolerable accuracy, and hence the great advantage of arithmetical calculations in the practical arts founded on geometry. By these the more complicated constructions of geometry are reduced to those simple operations which are found by experience to be capable of greatest exactness.

of a definition ; the simplicity of it, the ease with which the other properties of the figure may be derived from it, and sometimes even the ease with which it can be executed mechanically, may be considered as important circumstances.

In the straight line, the circle, the conic sections, and a few curves of the higher orders, the most convenient definitions, and the postulates connected with them, are generally known and received. An equation to a curve may also be assumed as a definition of it, and the description of it, according to that definition, may be considered as a postulate ; but, if the geometrical construction of problems is to be investigated by means of Algebra, it is often useful to deduce from the equation to a curve, those data, which, from the geometrical theory of the curve, are known to be necessary to its description in the original postulate, or in any problem founded upon it. This is called *constructing the locus* of an equation, and from this method are generally derived the most elegant constructions, which can be obtained by the use of algebra. In the following

lowing section, there is an example of a problem resolved by such constructions.

Sometimes a mechanical description of a curve line defined by an equation is useful; and, as the exhibition of it by such a motion as is supposed in that definition, is rarely practicable, it generally becomes necessary to contrive some more simple motion which may in effect correspond with the other, and may describe the curve with the degree of accuracy which is wanted. Frequently, indeed, the only method which can be conveniently practised, is the finding a number of points in the curve by the resolution of numeral equations, in the manner mentioned in Sect. I. of this Chapter, and then joining these points by the hand; and though this operation is manifestly imperfect, it is on some occasions useful.

II. *Solution of Problems.*

The solution of geometrical problems by algebra is much promoted, by describing the

the loci of the equations arising from these problems.

For this purpose, equations are to be derived, according to the methods formerly described, and then to be reduced to two, containing each the same two unknown quantities. The loci of these equations are to be described, the two unknown quantities being considered as the co-ordinates, and placed at the same angle in both. The co-ordinates at an intersection of the loci, will be common to both, and give a solution of the problem.

The simplicity of a construction obtained by this method, will depend upon a proper notation, and the choice of the equations of which the loci are to be described. These will frequently be different from what would be proper in a different method of solution.

P R O B.

P R O B. IV.

To find a Point F in the Base of the given Triangle ABC, so that the Sum of the Squares of FE, FD drawn from it perpendicular upon the two Sides, may be equal to a given Space.

Draw BH, CG perpendicular on the two sides, and let $FD = x$, $FE = y$, $BF = z$, $BC = b$, $BH = p$, $CG = r$, and the given space $FD^2 + FE^2 = m^2$.

From similar triangles $z : x :: b : r$
and $z = \frac{bx}{r}$.

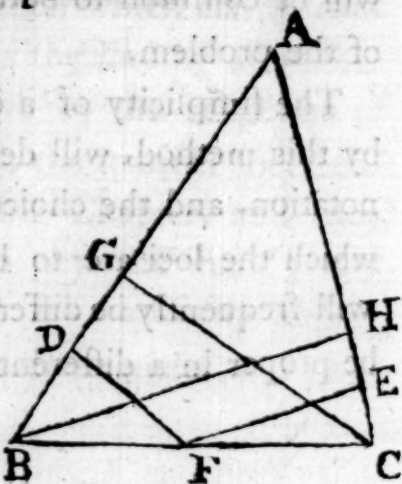
Also $b - z : y :: b : p$,

and $z = b - \frac{yb}{p}$

therefore $\frac{bx}{r} = b - \frac{yb}{p}$

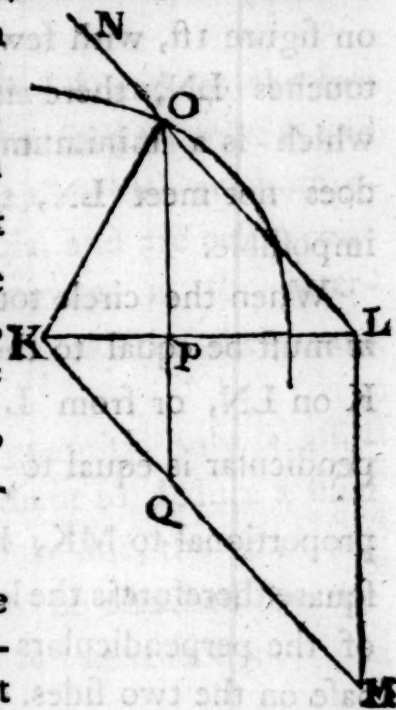
That is $y = p - \frac{px}{r}$, an equation to a straight line.

But



But $x^2 + y^2 = m^2$ of which the locus is a circle, having m for the radius. By constructing these loci, their intersection will give a solution of the problem.

Let $KL = CG (=r)$ be at right angles to $LM = BH (=p)$, join KM to which let LN be parallel; LN is the locus of the equation $y = p - \frac{px}{r}$; for, let any line OPQ be drawn parallel to LM , if $KP = x$, then $PQ = \frac{px}{r}$, and $QO = LM = p$, therefore $PO = y = p - \frac{px}{r}$.



About the centre K , with a distance equal to the line m , let a circle be described; that circle will be the locus of the equation $m^2 = x^2 + y^2$, for it is plain that if OP be any perpendicular from the circumference upon KL , KP being x , OP will be y . Either of the points therefore in which these two loci intersect each other, as O , will give OP an ordinate in both equations,

tions, KP being the common absciss ; therefore KP, OP are the two perpendiculars required, from which the point F is easily found.

The construction might have been made on figure 1st, with fewer lines. If the circle touches LN, there is only one solution which is a minimum ; and if the circle does not meet LN, the problem becomes impossible.

When the circle touches LN, the radius m must be equal to the perpendicular from K on LN, or from L on KM. This perpendicular is equal to $\frac{pr}{\sqrt{p^2+r^2}}$ or a fourth proportional to MK, KL, and LM, and its square therefore is the least sum of the squares of the perpendiculars from a point in the base on the two sides.

It may be remarked also, that the point which gives the sum of the squares a minimum, is found by dividing the base, in the proportion of the squares of the two sides of the triangle ; and this is easily demonstrated from the preceding construction.

P R O B.

P R O B. V.

Between two given Lines to find two mean Proportionals.

Let the lines be a and b , and let the two means be x and y ; therefore $a : x : y : b$, and hence $ay = x^2$, and $bx = y^2$, which are both equations to the parabola, and are easily constructed. The co-ordinates at the intersection of these two loci, will be the means required.

If one unknown quantity only is assumed, or if it is convenient to deduce a final equation containing only one, the construction of the roots is to be obtained by the method mentioned in the next section.

S C H O L I U M.

The constructions of the two preceding problems are geometrical; but it is some-

K k

times

times convenient to have a practical solution, by the mechanical description either of the algebraical lines employed in the geometrical solution, or of other geometrical lines by which it can be effected. But few of these are tolerably accurate, so that, in general, by means of calculation, the practical operations are all reduced to what may be performed by a ruler and a compass.

III. *Construction of Equations.*

The roots of an equation, containing only one unknown quantity, may be found by the intersection of lines, the product of whose dimensions is equal to the dimension of that equation. And hence problems are resolved without an algebraical solution of the equation arising from them.

Thus cubic and biquadratic equations may be constructed by the intersections of two conic sections as the circle and parabola, which are generally assumed as being most easily described.

In order to find these constructions, a new equation is to be assumed, containing two
variable

variable quantities, one of which is the unknown quantity of the given equation, and the other by substitution is to be inserted also in the given equation ; the intersection of the loci of these equations will exhibit the roots required.

Canons may be devised for the construction of particular orders, without assuming the new equation.

The final equation from prob. 5. would be $x^3 = a^2b$, which being constructed according to the rules, exhibits the common geometrical solution of that problem by the circle and parabola.

If an equation be assumed, as $ay = x^2$, the other by substitution becomes $xy = ab$; the locus of the former is a parabola, and of the latter an hyperbola, one of its asymptotes being the base, and the co-ordinates at their intersection will represent x and y ; the first of the two means is x , and in this case y is the other.

Equations also might be assumed so as to give a solution of this problem by other combinations of two of the conic sections, one of them not being the circle.

As

As geometrical magnitudes may be represented by algebra, so algebraical quantities and numbers may be represented by lines. Hence this construction of equations has sometimes been used as an easy method of approximation to the roots of numeral equations. For this purpose, the necessary straight lines must be laid down by means of a scale of equal parts, and the curve lines, on whose intersection the construction depends, must be actually described; the linear roots being measured on the scale, will give the numbers required. These operations may be performed with sufficient accuracy for certain purposes; but, as they depend on mechanical principles, the approximation obtained by them cannot be continued at pleasure; and hence it is seldom used, except in finding the first step of an approximation, which is to be carried on by other methods.

S C H O-

S C H O L I U M.

If the relation between the ordinate and absciss be fixed, but not expressible by a finite equation, the curve is called *Mechanical* * or *Transcendental*. This class is also sometimes defined by equations, by supposing either x or y in a finite equation, to be a curve line, of which the relation to a straight line cannot be expressed in finite terms.

If the variable quantities x or y enter the exponents of any term of an equation, the locus of that equation is called an *Exponential Curve*.

Many properties of these two classes of curves may be discovered from their equations.

* The term *Mechanical*, in this place, is used merely as the name of a particular class of curves, without implying that they have any more dependence on the principles of Mechanics or Physics, than the *algebraical* curves which have been treated of.

T H E E N D.